UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 1:30 pm to 3:30 pm

PAPER 347

ASTROPHYSICAL BLACK HOLES

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider the case where a supermassive black hole of mass $M_{\rm BH}$ is accreting dust particles each with a radius $r_{\rm d}$ and a density $\rho_{\rm d} = 1\,{\rm g\,cm^{-3}}$. Calculate the critical dust particle radius $r_{\rm d,crit}$ such that the black hole is able to shine at a luminosity which equals the Eddington luminosity in the electron scattering regime.

If in reality dust particles have a range of radii encompassing $r_{\rm d,crit}$ explain what will happen to the dust if the black hole is always shining at the same luminosity. You may ignore here any wavelength dependence of dust opacity.

(b) Consider now a dusty gas shell with mass $M_{\rm sh}$ initially at a distance R_0 from a supermassive black hole whose mass is $M_{\rm BH}$ and UV luminosity is L. You may neglect any other source of gravity. Derive an expression for the Eddington limit of this system and show that there exist three characteristic Eddington ratios $\Gamma = \frac{L}{L_{\rm Edd}}$ which depend on the nature of the dust opacity, where $L_{\rm Edd}$ is the Eddington luminosity.

If the shell is expanding away from the black hole in a medium of negligible density solve for the velocity of the shell at infinity v_∞ assuming that $v(R_0) = v_0$, $R_0 < R_{\rm UV} < R_\infty$, where $R_{\rm UV}$ is the transparency radius in UV, to obtain

$$v_\infty^2 = v_0^2 + \frac{2GM_{\rm BH}}{R_0}\left[\Gamma_{ss}\frac{R_{\rm UV}}{R_0} + \Gamma_{\rm IR} - 1\right]\left(1 - \frac{R_0}{R_{\rm UV}}\right) + \frac{2GM_{\rm BH}}{R_{\rm UV}}\left(\Gamma_{\rm UV} - 1\right),$$

where $\Gamma_{\rm IR}$, $\Gamma_{\rm ss}$ and $\Gamma_{\rm UV}$ are the Eddington ratios in IR optically thick, UV optically thick (single scattering) and UV optically thin limits, respectively.

If $R_{\rm UV} \gg R_0$ and $\Gamma_{\rm IR} \ll \Gamma_{\rm ss}\frac{R_{\rm UV}}{R_0}$ show that a large optical depth in UV leads to v_∞ in excess of escape velocity evaluated at R_0 for $L \sim L_{\rm Edd}$. What is the physical meaning of this result?

2 (a) Consider the case of a steady, spherically symmetric gas accretion onto a supermassive black hole of mass $M_{\rm BH}$ where the gas is at rest at infinity. Assume that the gas has a polytropic equation of state $p = K\rho^{1+1/n}$ and that the Bernoulli's constant is $H = u^2/2 + (n+1)K\rho^{1/n} - GM_{\rm BH}/r$, where p is the gas pressure, ρ is the gas density, K is a constant, n is the polytropic index, u is the gas velocity, G is the gravitational constant and r is the distance from the black hole. Derive an expression for the black hole accretion rate expressed as a function of gas density and sound speed at infinity.

If the gas equation of state is adiabatic with an adiabatic index $\gamma = 5/3$ show where the sonic transition occurs. Is it possible for this gas to accrete onto a black hole and why? What would be the correct expression for the accretion rate in this case?

(b) A singular isothermal halo with a velocity dispersion σ contains $f_{\rm gas}$ fraction of its mass in gas and a supermassive black hole with a mass $M_{\rm BH}$ approximatively on the observed scaling relation, i.e.

$$M_{\rm BH} \sim 3 \times 10^8 \,{\rm M_\odot} \left(\frac{\sigma}{200{\rm km\,s^{-1}}}\right)^4 .$$

In the very optimistic case of all gas within some radius R being able to accrete onto the black hole on a dynamical time scale show that the estimated accretion rate is less than 100 times the Eddington rate for $\sigma \sim 200{\rm km\,s^{-1}}$.

Explain why this is a very optimistic accretion rate and what in general we can deduce about the actual black hole fuelling efficiency.

Suddenly this gas is heated to a high temperature due to a burst of AGN feedback such that the gas sound speed $c_{\rm s,feed} \sim 10\sigma$. Estimate the likely change in the black hole accretion rate and comment on your result.

(c) A supermassive black hole with mass $M_{\rm BH}$ is subject to dynamical friction as it moves supersonically through a "sea" of star particles with an average velocity dispersion σ_*. Assume the black hole's velocity is $v \gg \sigma_*$, all stars have the same mass $m \ll M_{\rm BH}$ and the hole is non-rotating. Consider a single star with an impact parameter b and the minimum approach radius $r_{\rm min}$ to roughly derive that the star will be accreted by the black hole if

$$b < \frac{2GM_{\rm BH}}{cv} .$$

[*Hint: Use the conservation of energy and angular momentum.*]

Show that the accretion rate onto the black hole will be suppressed by a factor of $(v/c)^2$ compared to the gas accretion in Bondi-Hoyle-Lyttleton regime if the gas and stellar densities are comparable. What does this imply for the growth of supermassive black holes by stellar or dark matter capture?

3 (a) A Shakura-Sunyaev disc is steadily accreting at a rate $\dot{m}$ onto a supermassive black hole with mass $M_{\rm BH}$ where $\dot{m}$ is comparable to the Eddington rate. Qualitatively describe which spatial regions exist in the disc governed by different sources of opacity and pressure.

Viscous dissipation per unit disc face area is given by

$$\frac{1}{2}\nu\Sigma R^2\left(\frac{d\Omega}{dR}\right)^2 \simeq \frac{3GM_{\rm BH}\dot{m}}{8\pi R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right],$$

where R is the radial distance in cylindrical-polar coordinates, ν is the kinematic viscosity, Σ is the gas surface density, Ω is the gas angular velocity, and R_* corresponds to the innermost stable circular orbit. Derive an expression for the disc height valid in the innermost regions and roughly sketch the disc height as a function of R.

(b) Recalling that the thermal instability criterion can be expressed as

$$\frac{d\dot{Q}}{dT_{\rm c}} < 0\,,$$

where $\dot{Q}$ is the net cooling rate and $T_{\rm c}$ is the midplane temperature, derive whether the Shakura-Sunyaev disc is thermally unstable in these three scenarios: (i) the gas temperature is larger than 10^4 K and the gas is optically thin, (ii) the gas is optically thick and the main source of opacity is the Kramers' opacity, and (iii) the gas is optically thick and the main source of opacity is electron scattering. In all cases assume gas pressure dominates the total pressure.

(c) Assume now that $M_{\rm BH} = 10^8\,{\rm M}_\odot$ and that $\dot{m} = 0.1{\rm M}_\odot\,{\rm yr}^{-1}$. If the spin of the black hole is initially negligible roughly estimate the rate at which the angular momentum of the black hole increases due to gas accretion provided that the gas is always co-rotating. How long does it take to spin the black hole up to the maximally rotating Kerr black hole? By how much does the black hole mass increase in this time? What do you deduce from this calculation about the likelihood of low spinning black holes if they acquire most of their mass through a Shakura-Sunyaev disc? Describe if there is any (current or future) observational measurement that can constrain the average spin value of the entire black hole population?

END OF PAPER