MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2019 9:00 am to 11:00 am

PAPER 346

GALAXY FORMATION

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Derive the expression for the comoving Jeans length of a collisional fluid with homogeneous density $\overline{\rho}$ and sound speed c_s in the form

$$\lambda_J^{com} = \frac{c_s}{a(t)}\sqrt{\frac{\pi}{G\,\overline{\rho}}},$$

where $a(t)$ is the scalefactor.

What is the fate of perturbations with wavelength $\lambda < \lambda_J^{com}$?

If the fluid is collisionless, how does the formula for the comoving Jeans length change?

Suppose the Universe contains baryons and photons only. On a graph of comoving scale versus time, show the evolution of the comoving Jeans length for adiabatic, isentropic perturbations of baryons in an Einstein-de Sitter Universe. The plot should distinguish the behaviour of λ_J^{com} for the baryons from before the epoch of matter-radiation equality t_{eq} to after the epoch of recombination t_{rec}.

Now suppose the Universe contains cold dark matter particles which can be treated as a collisionless fluid. On a new plot, show the behaviour of the comoving Jeans length for cold dark matter particles. This should distinguish the behaviour of λ_J^{com} in the epochs when the cold dark matter particles are relativistic $t < t_{NR}$, are non-relativistic but still coupled to photons $t_{NR} < t < t_{dec}$ and are fully decoupled $t > t_{dec}$. The behaviour of λ_J^{com} should also be traced through the epochs of matter-radiation equality t_{eq} to after the epoch of recombination t_{rec}.

(b) In a flat Universe with non-zero cosmological constant Λ, the equation of motion of the radius of a shell enclosing mass M in a spherically symmetric perturbation is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM}{r^2} + \frac{\Lambda}{3} r.$$

Show that the evolution conserves energy E

$$E = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{GM}{r} - \frac{\Lambda}{6} r^2.$$

Assuming a uniform sphere with mass M and turnaround radius r_{ta}, show that the potential energies at turnaround due to gravity and Λ are

$$W_{G,ta} = -\frac{3}{5}\frac{GM^2}{r_{ta}}, \qquad W_{\Lambda,ta} = -\frac{1}{10}\Lambda M {r_{ta}}^2.$$

Show that, in the final virialized state, the kinetic and potential energies are related by

$$2T_f + W_{G,f} = 2W_{\Lambda,f},$$

where T_f is the final kinetic energy.

Hence, show that the final radius r_f satisfies the cubic equation

$$2\eta(r_f/r_{ta})^3 - (2+\eta)(r_f/r_{ta}) + 1 = 0,$$

with $\eta = \Lambda/(4\pi G\rho_{ta})$ where ρ_{ta} is the mean density at turnaround.

[QUESTION CONTINUES ON THE NEXT PAGE]

By making a suitable approximation, show further that

$$\frac{r_{\rm f}}{r_{\rm ta}} = \frac{1-\eta/2}{2-\eta/2}.$$

Provide a physical interpretation of this result, in particular comparing the repulsive or positive Λ case with the zero Λ case.

2 Suppose that any encounter of a galaxy with another galaxy leads to a merger if the impact parameter is less than R. Ignoring gravitational focussing, explain why the probability of a merger in time T is

$$P = \pi R^2 \langle v_{\rm rel} \rangle N T,$$

where N is the number density of galaxies and $\langle v_{\rm rel} \rangle$ is the mean relative velocity.

By estimating R, N and $\langle v_{\rm rel} \rangle$ and recalling that 1 kpc $\approx 3 \times 10^{16}$ km, compute an order of magnitude estimate for the probability of a merger in a Hubble time.

Consider a perturbing mass $m_{\rm p}$ which passes a galaxy at impact parameter $\underline{p} = p\hat{\underline{x}}$ with a large velocity $\underline{v} = v\hat{\underline{z}}$. Here, $\hat{\underline{x}}$ and $\hat{\underline{z}}$ are unit vector in the x-direction and the z-direction respectively. Suppose the centre of the galaxy is at the origin and the perturbing mass is in the (x, y) plane with position vector $\underline{r}_p$. For $r = |\underline{r}| < p$, show that the perturbing or tidal potential is

$$\psi(r) = \frac{G m_{\rm p}}{{r_{\rm p}}^3} \left[-\frac{r^2}{2} + \frac{3(\underline{r} \cdot \underline{r}_{\rm p})^2}{2{r_{\rm p}}^2} \right].$$

Here, we are using the convention that the tidal force $= \nabla\psi$, whilst $r_{\rm p} = |\underline{r}_{\rm p}|$ is the distance of the perturber.

Hence, in the impulse approximation, show that a star at $\underline{r}$ receives a velocity increment

$$\Delta \underline{v} = \frac{2 G m_{\rm p}}{v p^2} \left(x, -y, 0 \right).$$

Assuming $\Delta\underline{v}$ is uncorrelated with $\underline{v}$, show that the change in energy per unit mass is

$$\Delta E = \frac{2 G^2 {m_{\rm p}}^2}{v^2 p^4} (x^2 + y^2).$$

On averaging over a spherical galaxy of mass $m_{\rm g}$, deduce that

$$\Delta E = \frac{4 G^2 {m_{\rm p}}^2 m_{\rm g}}{3 v^2 p^4} \langle r^2 \rangle.$$

where $\langle r^2 \rangle$ is the mean square radius of stars in the galaxy.

Hence, deduce that in an encounter of two equal mass galaxies, the energy change is

$$\Delta E = \frac{8 G^2 {m_{\rm g}}^3}{3 v^2 p^4} \langle r^2 \rangle.$$

Explain why the orbital energy is

$$E = \frac{1}{4} m_{\rm g} v^2.$$

Show that the condition for a merger is

$$p v < \left[\frac{32}{3} G^2 {m_{\rm g}}^2 \langle r^2 \rangle \right]^{1/4}.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

Explain why this formula fails when $v/p < \Omega$, where Ω is the circling frequency of stars in the galaxy.

3 In the Zel'dovich approximation, how is the initial comoving position of a mass element $\underline{x}_\mathrm{i}$ at time t_i related to its present position $\underline{x}(t)$?

Stating clearly any assumptions, derive the Zel'dovich approximation in the form

$$\dot{\underline{x}} = -\frac{\dot{D}(t)}{4\pi G\,\overline{\rho}_\mathrm{m} a^3}\nabla\Phi_\mathrm{i},$$

where a is the scale factor (normalised to unity at the initial time t_i), $\overline{\rho}_\mathrm{m}(t)$ is the mean matter density, $D(t)$ is the linear growth rate and Φ_i is the potential perturbation at t_i. (Note that, throughout this question, the roman subscript 'i' designates 'initial' and is not a tensorial index).

What is the physical origin of the spin of a dark halo?

Let the Lagrangian region occupied by a virialized dark halo be V_L. The angular momentum of the material at early times is

$$\underline{J} = \int_{V_\mathrm{L}} \mathrm{d}^3 x_\mathrm{i}\,\overline{\rho}_\mathrm{m} a^3 (a\underline{x} - a\hat{\underline{x}}) \times \underline{v},$$

where $\hat{\underline{x}}$ is the barycentre of the volume and $\underline{v}$ is the peculiar velocity. Show that, to lowest order,

$$\underline{J} = -\frac{\dot{D}}{4\pi G\,\overline{\rho}_\mathrm{m} a^2}\int_{\Sigma_\mathrm{L}} \Phi_\mathrm{i}(\underline{x}_\mathrm{i})\,(\underline{x}_\mathrm{i} - \hat{\underline{x}}_\mathrm{i}) \times \mathrm{d}\underline{S},$$

where Σ_L is the surface bounding the volume V_L.

Show that the angular momentum vanishes if the volume V_L is spherical.

Evaluate the angular momentum if the bounding surface Σ_L is an equipotential.

By expanding Φ_i in a Taylor series, show

$$J_j = -\frac{\dot{D}}{4\pi G\,\overline{\rho}_\mathrm{m} a}\,\epsilon_{jkl} T_{km} I_{ml}, \qquad (1)$$

where ϵ_{jkl} is the completely antisymmetric tensor, I_{jk} is the moment of inertia tensor

$$I_{jk} = \int_{V_\mathrm{L}} \mathrm{d}^3 x_\mathrm{i}\,\overline{\rho}_\mathrm{m} a^3 (x_{\mathrm{i},j} - \hat{x}_{\mathrm{i},j})(x_{\mathrm{i},k} - \hat{x}_{\mathrm{i},k}),$$

and T_{jk} is a suitably defined tensor.

Compute how the angular momentum of a halo scales with time in an Einstein-de Sitter Universe.

When compared against numerical cosmological simulations of the build-up of structure, the angular momentum growth of halos is only roughly consistent with eq (1). State two reasons for this failure.

END OF PAPER