MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 9:00 am to 12:00 pm

PAPER 345

FLUID DYNAMICS OF THE ENVIRONMENT

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an inviscid incompressible two-dimensional flow with mean velocity field $\mathbf{U} = (U(z), 0)$ and non-diffusing Boussinesq background stratification characterised by the buoyancy frequency $N(z)$ about a reference density ρ_0. The coordinates $\mathbf{x} = (x, z)$ are oriented with z vertically upward.

(a) Linearise the equations of motion and hence derive an equation governing the vertical velocity fluctuations $w(x, z, t)$. Show that in the limit of a stationary disturbance this reduces to

$$\left(\nabla^2 + \frac{N^2}{U^2} - \frac{U''}{U}\right) w = 0,$$

where $U'' = d^2U/dz^2$. What restriction(s) must be applied to U and N?

(b) Consider an upward propagating stationary disturbance originating at $z = 0$ with wavenumber vector $\mathbf{k} = (k, m)$. Outline the conditions required for the WKB approximation to be applied. Consider the case $U/N = (1 + \gamma z)U_0/N_0$ with $|UU''| \ll N^2$ and γ a constant. Using the WKB approximation, sketch for both $\gamma < 0$ and $\gamma > 0$ (i) the direction of energy propagation, (ii) the orientation of the wavenumber vector, and (iii) the orientation and spacing of lines of constant phase. In both cases, determine the height where the character of the waves changes qualitatively. Discuss briefly whether the WKB approximation is uniformly valid.

(c) Consider now the case of the mean velocity and continuous background stratification given by

$$U(z) = \begin{cases} U_2 & z \geqslant 0, \\ 0 & z < 0, \end{cases} \qquad N(z) = \begin{cases} N_2 & z \geqslant 0, \\ N_1 & z < 0, \end{cases}$$

for constant U_2, N_1 and N_2. An oscillatory disturbance located at $z \ll 0$ produces waves with amplitude $|\mathbf{u}_0| = A_0$, frequency $\omega_0 < N_1$ and wavenumber vector $\mathbf{k}_0 = (k_0, m_0)$. Assuming these waves propagate to the right and reach $z = 0$, use matching conditions to show that the amplitude A_t of the reflected wave is given by

$$\frac{A_t}{A_0} = \frac{2}{\frac{\cos\theta_2}{\cos\theta_1} + \left(1 - \frac{k_0 U_2}{\omega_0}\frac{\sin\theta_2}{\sin\theta_1}\right)}.$$

Give expressions for the angles θ_1 and θ_2. Suppose $N_1 = N_2$ and $\omega_0/N_1 = \frac{1}{2}$. Sketch the orientation of the lines of constant phase for the case when $k_0U_2 = -\frac{1}{4}N_1$. What happens when $k_0U_2 = \frac{1}{2}N_1$?

2

Consider a high Reynolds number flow in an expanding channel with a triangular cross-section. With z oriented vertically upward, the bottom of the channel is given by $z = 0$ at $y = 0$ for $x \geqslant 0$, and the top of the sloping side walls by $z = H$ at $y = \pm\beta x$. The fluid is of density ρ_0 throughout the system, but there is an interface at $z = h(x, t)$ (with $h \leqslant H$). The fluid below the interface is laden with particles of density ρ_p at a volume concentration $\phi(x, t)$. The particle-laden fluid has velocity $u(x, t)$, averaged over the triangular cross-section. Above the interface the fluid may be considered quiescent and devoid of particles.

(a) Specify an appropriate model for the settling of the particles near a horizontal boundary if the settling velocity of an individual particle in isolation is W_s. How is this affected by weak turbulent stirring? How might you expect the settling to be modified if the boundary is not horizontal? Give an expression for the reduced gravity g' of the particle-laden fluid.

(b) Under what conditions can the flow be considered as 'shallow water'? Assume that the flow is Boussinesq with $\phi \ll 1$ and there is weak stirring such that the volume of the layer is conserved. Formulate the shallow water equations in terms of h, u and ϕ for the flow in this channel. (You may assume the particle concentration $\phi(x, t)$ remains uniform in y and z below the interface and the settling is not affected by the sloping walls.) Show that the equations are hyperbolic with three characteristics and determine the corresponding ordinary differential equations for each characteristic.

(c) For $t < 0$ the particle-laden fluid is confined to the region $0 < x \leqslant L_0$ with $h = H$, $\phi = \phi_0 \ll 1$ and $u = 0$. At $t = 0$ the fluid is released to form a gravity current flowing along the channel in the positive x direction. Specify a suitable front condition and derive an integral model for the development of the gravity current. Determine the maximum distance along the channel the current can propagate.

3

Consider a vertical axisymmetric time-dependent turbulent buoyant plume of radius $b(z,t)$, density $\rho(z,t)$ and vertical velocity $w(z,t)$ in a quiescent homogeneous fluid of constant density ρ_0.

(a) Give an expression for 'Batchelor entrainment' into the plume under the assumption of a non-Boussinesq density difference. Define the buoyancy flux $F(z,t)$, mass flux $Q(z,t)$ and momentum flux $M(z,t)$ for 'top-hat' profiles and derive the 'plume equations' in terms of these quantities.

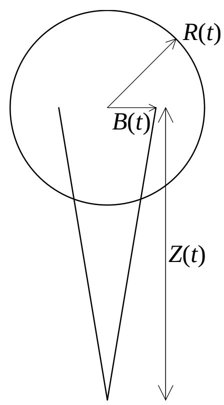
(b) Solve the plume equations for the case of a steady Boussinesq plume from a point source at $z = 0$ providing a buoyancy flux $F_0 > 0$ but zero mass and momentum fluxes. Determine corresponding expressions for b, ρ, w and the reduced gravity $g'(z)$. Show that the Froude number $C_p = w/\sqrt{g'b}$ is constant and determine its value. Discuss briefly the validity of the Boussinesq assumption in this case.

(c) Consider now a 'starting plume' from a point source at $z = 0$ where the buoyancy flux at the source is given by

$$F(0,t) = \begin{cases} 0 & t < 0, \\ F_0 & t \geqslant 0, \end{cases}$$

and $Q(0,t)$, $M(0,t)$ are zero. For $t > 0$ the starting plume can be modelled as the combination of a steady plume of width $b(z)$ for $z < Z(t)$ and a spherical 'thermal' with reduced gravity $g_t(t)$ and radius $R(t)$ that is centred on $z = Z(t)$ and rising with velocity $U(t)$ (see figure). For this model the only entrainment into the thermal is due to the flux at $z = Z(t)$ from the steady part of the plume. The radius of the thermal must be at least as large as $B(t) = b(Z)$, the radius of the plume at this height (i.e. $R \geqslant B$).

(i) If the starting plume is self-similar, what functional forms must R, U and g_t take?



(ii) Derive equations for conservation of volume and mass for the thermal and hence show that the reduced gravity in the thermal is given by

$$g_t = \frac{9}{4} g_p,$$

where $g_p = g'(Z)$ is the reduced gravity at the top of the steady part of the plume.

(iii) Show that the Froude number of the thermal $C_t = U/\sqrt{g_p R}$ is constant and derive an expression for C_t/C_p as a function of R/B. By considering the limits on R, show that $C_t/C_p < 10/(15 + 72\alpha)$.

END OF PAPER