MAT3, MAMA

**MATHEMATICAL TRIPOS Part III**

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Thursday, 6 June, 2019 1:30 pm to 4:30 pm

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# PAPER 341

# NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt **THREE** questions from Section A and **ONE** question from Section B.*

*Each question from Section B carries twice the weight of a question from Section A.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION A

**1**

Consider the numerical method

$$\mathbf{y}_{n+2} - \frac{8}{7}\mathbf{y}_{n+1} + \frac{1}{7}\mathbf{y}_n = \frac{6}{7}h\mathbf{f}(\mathbf{y}_{n+2}) - \frac{2}{7}h^2\frac{\partial \mathbf{f}(\mathbf{y}_{n+2})}{\partial \mathbf{y}}\mathbf{f}(\mathbf{y}_{n+2})$$

for the initial-value ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

(a) Determine the order of the method. Is it convergent?

(b) Is the method A-stable?

**2**

Consider the Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} .$$

(a) Determine the order of this method.

(b) Is the method A-stable?

(c) Is it algebraically stable?

**3**

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x}.$$

given, where $\alpha$ is a given real parameter.

(a) The equation is given with zero Dirichlet boundary conditions at $x = 0$ and $x = 1$ and semidiscretized with the scheme

$$u'_m = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{\alpha}{2\Delta x}(u_{m+1} - u_{m-1}), \qquad m = 1, \ldots, M,$$

where $\Delta x = 1/(M+1)$. Is it stable?

(b) The equation is given as a Cauchy problem and the scheme

$$u'_m = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{\alpha}{2\Delta x}(u_{m+1} - u_{m-1}), \qquad m \in \mathbb{Z},$$

is solved with forward Euler scheme. Write explicitly the fully discretized scheme and determine its stability for different values of $\Delta x$ and $\Delta t$.

**4**

The advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

given with an initial condition at $t = 0$ and a periodic boundary condition, is solved by the two-step fully-discretized scheme

$$u_m^{n+1} = (1 - 2\mu)(u_m^n - u_{m+1}^n) + u_{m+1}^{n-1}, \qquad \mu = \frac{\Delta t}{\Delta x}.$$

(a) Determine the order of this method.

(b) Carefully justifying your steps, find the range of the Courant numbers $\mu$ for which the scheme is stable.

**5**

The advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad x \in [0,1], \quad t \geqslant 0,$$

given with an initial value for $t = 0$ and with *periodic* boundary conditions at the endpoints, is discretized using the Galerkin method with chapeau functions.

(a) Derive explicitly the underlying semidiscretized scheme.

(b) Determine whether this scheme is stable.

## SECTION B

**6**

Write an essay on linear and nonlinear stability of Runge–Kutta methods. Your essay should include definitions, statements and their proofs, and examples.

**7**

Write an essay on the theory of Ritz and Galerkin finite element methods for boundary value problems. Your essay should include definitions, statements and their proofs, and examples. You may assume for simplicity zero Dirichlet boundary conditions throughout.

## END OF PAPER