UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2019 9:00 am to 11:00 am

PAPER 339

TOPICS IN CONVEX OPTIMISATION

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Given integers $1 \leqslant k \leqslant n$ define

$$P_k = \left\{ y \in \mathbb{R}^n : 0 \leqslant y_i \leqslant 1 \ \forall i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} y_i = k \right\}.$$

Show that the extreme points of P_k can only have integer coordinates in $\{0, 1\}$. [10]

For a vector $x \in \mathbb{R}^n$ define $f_k(x)$ to be the sum of the k largest components of x. For example $f_1(x) = \max_{i=1,\ldots,n} x_i$ and $f_n(x) = \sum_{i=1}^{n} x_i$.

(b) Show that $f_k(x)$ can be expressed as the solution of a linear maximization program with $2n$ inequality constraints and one equality constraint. The constraints of your linear program should not depend on x; i.e., only the objective function of your linear program can depend on x. [10]

(c) Use linear programming duality to give a minimization formulation of f_k. [10]

(d) Use the previous question to show that any optimization problem of the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f_k(x) \quad \text{subject to} \quad Ax = b$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ can be expressed as a linear program. [10]

(e) Repeat questions (b) and (c) with the function $f_k(x)$ replaced by $F_k(X)$ defined on $n \times n$ symmetric matrices $X \in \mathbf{S}^n$ as the sum of the k largest eigenvalues of X. Give maximization and minimization formulations of F_k using semidefinite programming. Only explain briefly the differences with respect to $f_k(x)$; you do not need to produce detailed proofs. [10]

2

Given a graph $G = (V, E)$ with vertex set V and edge set E, the (complement) theta number of G is defined as

$$\bar{\vartheta}(G) = \min \left\{ Z_{00} : \begin{bmatrix} Z_{00} & \mathbf{1}^T \\ \mathbf{1} & Z \end{bmatrix} \succeq 0,\ Z_{ii} = 1\ \forall i \in V,\ Z_{ij} = 0\ \forall ij \in E \right\}, \qquad (1)$$

where $\mathbf{1}$ is the vector of all-ones. In the following we will identify the vertex set V with $\{1, \dots, n\}$.

(a) Using Schur complements, show that (1) has the following alternative formulation:

$$\bar{\vartheta}(G) = \min \{ t : U_{ii} = t - 1\ \forall i \in V,\ U_{ij} = -1\ \forall ij \in E,\ U \succeq 0 \}. \qquad (2)$$

In the remaining questions we will see how to use the solution of $\bar{\vartheta}(G)$ to construct a (semi) *coloring* of G. Recall that a k-coloring of G is a map $c : V \to \{1, \dots, k\}$ such that if $ij \in E$ then $c(i) \neq c(j)$. A *k-semicoloring* is a map $c : V \to \{1, \dots, k\}$ that is a valid coloring on at least half the vertices of the graph, i.e., there is a subset W of V with $|W| \geqslant |V|/2$ such that for any $i, j \in W$ and $ij \in E$ we have $c(i) \neq c(j)$.

(b) Show that if G has a 3-coloring, then $\bar{\vartheta}(G) \leqslant 3$. [*Hint: define an appropriate mapping $d : V \to \{e_0, e_1, e_2\}$ where $e_0, e_1, e_2 \in \mathbb{C} \simeq \mathbb{R}^2$ are the third roots of unity and consider $U_{i,j} = 2\langle d(i), d(j)\rangle$.*] [10]

(c) Assume G has a 3-coloring and let U be a solution of the semidefinite program (2). We can factorize the matrix $\frac{U}{t-1}$ so that for all $i, j \in \{1, \dots, n\}$, $\frac{U_{ij}}{t-1} = \langle v_i, v_j \rangle$ where $v_1, \dots, v_n$ are unit vectors in $\mathbb{R}^p$ (i.e., $\|v_i\|_2^2 = 1$ for all i). Since G is 3-colorable, by the previous question we have, for $ij \in E$, $\langle v_i, v_j \rangle = -\frac{1}{t-1} \leqslant -\frac{1}{2}$.

(i) Consider a random hyperplane $H = \{x \in \mathbb{R}^p : \langle a, x \rangle = 0\}$ with a chosen uniformly at random from the standard normal distribution in $\mathbb{R}^p$. We say that the hyperplane H cuts the edge $ij \in E$ if v_i and v_j lie on two different sides of the hyperplane. Show that the probability that a random hyperplane H cuts an edge ij is at least $2/3$. [10]

(ii) We now take r random hyperplanes $H_1, \dots, H_r$ identically and independently distributed. The r hyperplanes $H_k = \{x \in \mathbb{R}^p : \langle a_k, x \rangle = 0\}$ $(k = 1, \dots, r)$ partition the space $\mathbb{R}^p$ into at most 2^r regions. We propose a 2^r-coloring of the vertices of the graph as follows: let $c : \{1, \dots, n\} \to \{-1, 1\}^r$ with $c(i) = (\text{sign}(\langle a_k, v_i \rangle))_{k=1,\dots,r}$. Prove that the expected number of edges that have the same color at their endpoints (i.e., the "bad" edges) is $\leqslant (1/3)^r m$ where $m = |E|$ is the number of edges, i.e., [10]

$$\mathbb{E}\Big[|\{ij \in E : c(i) = c(j)\}| \Big] \leqslant (1/3)^r m.$$

(iv) Using an appropriate choice of r show how to construct a (randomized) k-semicoloring with $k = O(n^\gamma)$ where $\gamma < 1$ is to be specified. [10]

3

An $n \times n$ symmetric matrix A is called *copositive* if $x^T A x \geqslant 0$ for all $x \in \mathbb{R}^n_+$ where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geqslant 0 \; \forall i = 1, \ldots, n\}$.

(a) Show that A is copositive if and only if the degree-four homogeneous polynomial

$$p(z_1, \ldots, z_n) = \sum_{1 \leqslant i,j \leqslant n} A_{ij} z_i^2 z_j^2.$$

is globally nonnegative. [10]

(b) Show that the polynomial $p(z)$ is a sum-of-squares if and only if A can be written as $A = P + N$ where P is a symmetric positive semidefinite matrix and N is a symmetric matrix whose entries are all nonnegative. [15]

(c) Consider the matrix

$$H = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

By observing that

$$x^T H x = (x_1 - x_2 + x_3 - x_4 + x_5)^2 + 4x_2x_5 + 4x_1(x_4 - x_5)$$

show that H is copositive. [10]

(d) Show that H cannot be written as $H = P + N$ where P is positive semidefinite and N is elementwise nonnegative. [*Hint: consider* $x^T H x$ *where* $x = (1, 2, 1, 0, 0)^T$]. [15]

END OF PAPER