UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

**MATHEMATICAL TRIPOS Part III**

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Thursday, 30 May, 2019 9:00 am to 11:00 am

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# PAPER 338

# OPTICAL AND INFRARED ASTRONOMICAL TELESCOPES AND INSTRUMENTS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## 1 Spectrographs, apparent magnitudes and the Earth's atmosphere

(a) A plane wavefront of light with wavelength $\lambda$ is incident on a plane diffraction grating with an angle $\alpha$ which is measured from the normal to the grating. The light is diffracted off the grating with an intensity $I(\beta)$ where $\beta$ is an angle also measured from the grating normal. The grating has grooves which are regularly spaced with a distance between grooves of $d$. The intensity distribution $I(\beta)$ has a pattern of distinct bright peaks and each peak corresponds to an integer $m$ which is called the spectral order.

(i) What physical condition gives rise to the intensity peaks? Use your answer and draw a diagram to derive the function $g(\alpha, \beta, \lambda, d, m) = 0$ (i.e. the grating equation).

(ii) Now consider polychromatic light. The diffracted light is captured by a camera of focal length $f_{cam}$. Use your grating equation to derive an expression for the dispersion $q$ in the focal plane of the camera. The units of $q$ are distance in the focal plane per unit wavelength.

(iii) The spectral resolution is $R = \lambda/\Delta\lambda$ where $\Delta\lambda$ is the wavelength separation for two spectral features that are only just resolved. If the light is delivered to the grating through a slit of physical width $x$ and a collimator of focal length $f_{coll}$ show that

$$R = \frac{m f_{coll} \lambda}{x d \cos\beta}.$$

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(b) A cross-dispersed echelle spectrograph has a flat focal plane. In this plane the y-axis is a line through the camera's optical axis which is orthogonal to the main dispersion direction of the echelle grating. For a single spectral order, there is a range of wavelengths, $S$, that arrive at the detector closer to the y-axis than they do in any of the other orders. This is called the free spectral range and the detector width must be at least equal to its physical length for all spectral orders to ensure that there are no spectral coverage gaps in the echellogram.

(i) Draw a diagram, in the focal plane of the spectrograph, showing two spectral orders with order $m$ and $(m+1)$ which are separated on the detector by a cross-disperser. Indicate on your diagram the directions of increasing wavelength for both the main (echelle grating) dispersion and the cross-dispersion.

(ii) Show that

$$S = \frac{K}{m(m+1)}$$

where $K$ is a constant. What is the physical meaning of the constant $K$?

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**[QUESTION CONTINUES ON THE NEXT PAGE]**

(c) The intensity of light from a star is measured in a passband with $\lambda_1 < \lambda < \lambda_2$ as

$$I = \int_{\lambda_1}^{\lambda_2} \eta_\lambda F_\lambda d\lambda$$

where $F_\lambda$ is the flux at wavelength $\lambda$ and $\eta_\lambda$ is the instrumental efficiency at $\lambda$.

(i) Write down an equation for the apparent magnitude of the star based on the measurement $I$ and a reference measurement $I_0$ that defines the zero point of the magnitude system.

(ii) In the UBV photometric system apparent magnitudes are designated by capital letters according to defined passbands. For example, the B magnitude is defined for a central wavelength of 440nm. What are the designating letters and approximate central wavelengths of the filters in this system for wavelengths between 1.0$\mu$m and 5.0$\mu$m? What practical consideration led to these passbands being chosen?

(iii) $N$ objects with a uniform space density and identical intrinsic brightness can be seen in the whole sky for a magnitude limit of $m_1$. What magnitude limit, $m_{10}$, do we need to reach to observe $10N$ of these objects?

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(d) Describe all the various ways in which the Earth's atmosphere changes the observed properties of wavefronts of light from a distant object such as a star.

## 2 Adaptive optics and high contrast imaging

(a) A ground-based telescope has an adaptive optics (AO) system with a deformable mirror (DM) as the phase corrector.

(i) What is the purpose of an AO system?

(ii) Sketch an AO system showing all the main components and the optical paths connecting them.

(iii) Explain what is meant by the term *Strehl ratio.*

(iv) Discuss sky coverage for an AO system. Why cannot an AO system be used at any point on the sky?

(v) For a ground-based observatory, what is meant by the term *seeing* and what units is it normally expressed in?

(vi) How is seeing related to the Fried Parameter, $r_0$? If $r_0 \propto \lambda^{6/5}$ how does the number of DM actuators scale with $\lambda$? How does the seeing scale with $\lambda$?

(vii) A point on a unit disc is described by a radial coordinate $\rho$ and an azimuthal coordinate, $\phi$. $\rho = 1$ at the edge of the disc. A wavefront error is described by an even Zernicke polynomial

$$Z_n^m(\rho, \phi) = R_n^m(\rho) \cos m\phi$$

where $n$ is the radial degree and $m$ is the azimuthal degree and $R_n^m(\rho)$ is a radial polynomial given by

$$R_n^m(\rho) = \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^k (n-k)!}{k!(\frac{n+m}{2} - k)!(\frac{n-m}{2} - k)!} \rho^{n-2k}$$

Derive $Z_4^0(\rho, \phi)$ and sketch this for a chord across the unit disc passing through the centre. Which Seidel aberration does this correspond to?

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(b) A high contrast imaging system has a DM with $N \times N$ actuators in a square format. The telescope has a circular primary mirror of diameter $D$ which is imaged on to the DM such that $N$ actuators span the diameter of the telescope. The total wavefront error can be thought of as the sum of many sinusoidal ripples in the wavefront each of which can be characterised by its amplitude, phase and wavelength, $\Lambda$ (which should not be confused with the wavelength of the light).

(i) What is the shortest spatial scale of the wavefront error (measured at the primary mirror) that the DM can correct?

(ii) Let a single wavefront error mode be one for which $D = j\Lambda/2$ where $j$ is an integer. Show that the number of modes that can be corrected, $M$, for the full telescope aperture is given by

$$M = \frac{\pi}{4} \left[ \frac{N}{2} \right]^2.$$

**[QUESTION CONTINUES ON THE NEXT PAGE]**

(iii) The system is used to observe a star. The intensity of a speckle of starlight produced by a single mode of the residual wavefront error (i.e. after the DM) is given by

$$C \equiv \frac{I_{spec}}{I_{star}} = (\pi h_0/\lambda)^2$$

where $\lambda$ is the wavelength of the light, $h_0$ ($\ll \lambda$) is the amplitude of the mode that gives rise to the speckle and $I_{star}$ is the total intensity of the light from the star. This equation also defines the speckle contrast $C$. Show that

$$h_{rms} = \frac{N\lambda\sqrt{C}}{4\sqrt{\pi}}$$

where $h_{rms}$ is the rms amplitude of the total wavefront error due to all the modes combined.

(iv) Show that the DM can correct wavefront errors and therefore improve the speckle contrast within an angle $\theta$ from the star (measured in the direction of a row or column of actuators) given by

$$\theta = \frac{N\lambda}{2D}.$$

Comment on the shape of the corrected region and its size in terms of spatially resolved elements.

## 3 Optics and multi-object spectrographs

(a) If a pair of conjugate object and image points (S and P), for a lens with focal length $f$, are separated by a distance $L > 4f$ there will two locations of the lens, a distance $d$ apart, for which the two real images can be observed. Show that

$$f = \frac{L^2 - d^2}{4L}.$$

Comment on the practical use of this result.

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(b) A cemented achromatic doublet lens has zero axial chromatic aberation for a pair of specific wavelengths, one in the red part of the spectrum (the Fraunhofer C line) and the other in the blue part of the spectrum (the Fraunhofer F line). The focal length, $f$, of a singlet lens is given by

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_a} - \frac{1}{R_b}\right) \equiv (n-1)\rho$$

where $n$ is the refractive index of the lens and $R_a$ and $R_b$ are the radii of curvature of the two lens surfaces (which define the quantity $\rho$).

(i) Show that

$$\frac{\rho_1}{\rho_2} = -\frac{n_{2F} - n_{2C}}{n_{1F} - n_{1C}}$$

where the subscripts 1 and 2 refer to the two lenses of the doublet and subscripts C and F refer to the red and blue wavelengths.

(ii) Now consider a third wavelength which is in between the red and blue wavelengths at the Fraunhofer d line. Show that

$$\frac{f_{2d}}{f_{1d}} = -\frac{(n_{2F} - n_{2C})/(n_{2d} - 1)}{(n_{1F} - n_{1C})/(n_{1d} - 1)}.$$

(iii) Using the definition

$$V_d \equiv \frac{n_d - 1}{n_F - n_C}$$

show that

$$f_{1d} = \frac{f_d(V_{1d} - V_{2d})}{V_{1d}} \quad \text{and} \quad f_{2d} = \frac{f_d(V_{2d} - V_{1d})}{V_{2d}}$$

where $f_d$ is the focal length of the doublet at the d line wavelength.

(iv) What is the quantity $V_d$ called? Draw a graph of $n_d$ versus $V_d$ with your axes spanning values that are typical for optical glasses. Indicate the areas where available glasses lie and which areas correspond to crown and flint glasses. What do the $V_d$ values imply for the focal lengths of the two lenses in the doublet?

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**[QUESTION CONTINUES ON THE NEXT PAGE]**

(c) A light ray passes through a prism (of refractive index $n$) which is emersed in air ($n = 1$). The ray hits a surface of the prism with an angle $\theta_1$ and emerges into the prism with an angle $\theta_2$. It then hits another surface of the prism with an angle $\theta_3$ and emerges in to the air at an angle $\theta_4$. These ray angles are measured with respect to the surface normal. The total deviation of the ray is $\gamma$ and the angle between the two prism surfaces is $\phi$.

(i) Show mathematically that minimum deviation occurs when the geometry of the system is symmetrical.

(ii) Make a physical argument for why minimum deviation occurs when the geometry of the system is symmetrical.

(iii) Derive an expression for $n$ in terms of the apex angle $\phi$ and the minimum deviation angle $\gamma_{min}$ and comment on the practical use of this result.

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(d) (i) What are the two main types of multi-object spectrographs?

(ii) Explain how they work.

(iii) Compare the two (giving reasons) in terms of field of view on the sky, spectral resolution and faintness limit.

## END OF PAPER