UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2019 9:00 am to 11:00 am

PAPER 336

PERTURBATION METHODS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Find the first two nonzero terms in the asymptotic expansions of the roots of

$$x\exp(1/x) = \exp(u)$$

for large positive u.

(b) The function $A_\nu(a\nu)$ is defined by

$$A_\nu(a\nu) = \int_0^\infty \exp(-\nu(a\sinh t - t))\mathrm{d}t \ ,$$

where a and ν are real.

Calculate the leading term in the asymptotic expansion of $A_\nu(a\nu)$ as $\nu \to \infty$ through positive real values with a held fixed in each of the following cases:

(i) $a > 1$;

(ii) $0 < a < 1$;

(iii) $a = 1$.

[In (iii) you may wish to express your answer in terms of the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)\mathrm{d}t \ . \qquad]$$

Show that

$$A_\nu(\nu + l\nu^{1/3}) \sim 2^{1/3}\nu^{-1/3}I(-2^{1/3}l) \tag{1}$$

for fixed l and large positive ν, where

$$I(x) \equiv \int_0^\infty \exp(xt - t^3/3)\mathrm{d}t \ .$$

Demonstrate explicitly that, for appropriate limiting values of l, equation (1) is consistent with each of your answers to (i), (ii) and (iii) above.

2 The function $f(r;\varepsilon)$ satisfies the equation

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \varepsilon^2 f = \varepsilon^2 r f^3 \quad \text{in} \quad r \geqslant 1\,,$$

where $0 < \varepsilon \ll 1$. The function $f(r;\varepsilon)$ also satisfies the boundary conditions

$$f = 1 \quad \text{on} \quad r = 1\,, \quad \text{and} \quad f \to 0 \quad \text{as} \quad r \to \infty\,.$$

Solve for f by obtaining asymptotic expansions up to and including terms of $O(\varepsilon^2)$, in two asymptotic regions, $r = O(1)$ and $r = O(\varepsilon^{-\beta})$, where β is a constant to be identified.

Explain how to contruct a composite expansion from the two asymptotic expansions.

Hints. You may quote the general solution, $y(x)$, of

$$\frac{d^2 y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - y = \frac{e^{-3x}}{x^2}\,,$$

with $y \to 0$ as $x \to \infty$, as

$$y = \alpha\frac{e^{-x}}{x} + \frac{1}{2x}\int_x^\infty \frac{e^{-x-2t} - e^{x-4t}}{t}\,dt\,,$$

where α is a constant. The limit of this solution as $x \to 0$ is given by

$$y = \frac{\alpha + \frac{1}{2}\ln 2}{x} + \ln x - \alpha + \tfrac{3}{2}\ln 2 + \gamma - 1 + O(x \ln x)\,,$$

where γ is Euler's constant.

3

(a) The function $\psi(x,t)$ satisfies the partial differential equation

$$5\psi_{tt} + \psi_{xxxx} + 4\psi = \varepsilon\psi\psi_x \,,$$

where $0 < \varepsilon \ll 1$. Seek solutions of the form

$$\psi = A(T)e^{ik_1(x-ct)} + B(T)e^{ik_2(x-ct)} + A^*(T)e^{-ik_1(x-ct)} + B^*(T)e^{-ik_2(x-ct)} \,,$$

where k_1, k_2 and c are positive real constants, T is an appropriate slow-time scale, and * denotes a complex conjugate.

Identify $k_1 < k_2$ and c such that the weak quadratic interaction induces a resonance; in doing so identify the appropriate slow-time scale over which this resonance develops. For the identified values of k_1, k_2 and c, find governing equations for the complex amplitudes A_0 and B_0 respectively, where A_0 and B_0 are the leading terms of asymptotic expansions of A and B.

Write

$$A_0(T) = R(T)e^{i\theta(T)} \quad \text{and} \quad B_0(T) = P(T)e^{i\varphi(T)} \,,$$

where the amplitudes R and P and the phases θ and φ are real. By considering a second-order equation for A_0 that is independent of B_0, or otherwise, obtain governing equations for the real amplitudes and phases, and deduce that

$$(R^2\theta_T)_T = 0 \quad \text{and} \quad (P^2\varphi_T)_T = 0 \,.$$

On the assumptions that the phases are constant, that $\varphi = 2\theta + \pi$, and that $R(0) = 4$ and $P(0) = 2$, solve for R and P, and briefly comment on the solution.

Hints. Solve for P first; the solution to

$$2P_T = P^2 + \gamma^2 \quad \text{is} \quad P = \gamma \tan\left(\tfrac{1}{2}\gamma(T+T_0)\right) \,,$$

where γ and T_0 are real constants.

(b) The function $\Psi(x,t)$ satisfies the partial differential equation

$$5\Psi_{tt} + \Psi_{xxxx} + 4\Psi = \varepsilon\Psi^2 \,,$$

where $0 < \varepsilon \ll 1$. For given real $\kappa > 0$, seek constants $\alpha(\kappa)$ and $\beta(\kappa)$ such that periodic solutions of the form

$$\Psi = \alpha\cos\left((1+\varepsilon\kappa)x - t\right) + \beta\cos 2\left((1+\varepsilon\kappa)x - t\right)$$

exist.

END OF PAPER