UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

# MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2019 1:30 pm to 3:30 pm

## PAPER 335

## DIRECT AND INVERSE SCATTERING OF WAVES

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Consider the three-dimensional problem of a time-harmonic source $Q(\mathbf{r})$ enclosed in the volume $V$ outside the closed and continuous surface $S = \partial D$, where $D$ is a scatterer.

The field $\psi$ generated by the source $Q(\mathbf{r})$, satisfies the Helmholtz equation

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = -Q(\mathbf{r}) \ , \tag{1}$$

with Sommerfeld boundary conditions at infinity, and some unknown boundary conditions on $\partial D$.

(i) Using the divergence theorem or otherwise, derive the Kirchhoff-Helmholtz equation. Hence write an expression for the scattered field $\psi_s(\mathbf{r})$ in $\mathbb{R}^3_{\setminus D}$.

(ii) Now assume that the incident field generated by the source is a plane wave $\psi_i(\mathbf{r}) = e^{ik\mathbf{r}\cdot\hat{\mathbf{r}}_0}$, in the direction $\hat{\mathbf{r}}_0$, and that the far field is measured and known.

Write an expression for the far field $f_\infty(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0, k)$ associated with $\psi_s(\mathbf{r})$ in terms of an integral over the surface of the scatterer $\partial D$.

Hence, given the function

$$v_g(\mathbf{r}') = \int_{S_1} e^{ik\hat{\mathbf{r}}\cdot\mathbf{r}'} g(\hat{\mathbf{r}})ds(\hat{\mathbf{r}}) \ , \qquad \mathbf{r}' \in \mathbb{R}^3 \ , \tag{2}$$

where $\int_{S_1} \ldots ds(\hat{\mathbf{r}})$ denotes integration over the surface of the unit ball $S_1$, and the function $g(\hat{\mathbf{r}}) \in L^2(S_1)$, write an expression for

$$\int_{S_1} f_\infty(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0, k) g^*(\hat{\mathbf{r}})ds(\hat{\mathbf{r}}) \qquad \text{(where } g^* \text{ denotes the complex conjugate)} \tag{3}$$

in terms of the unknowns $\psi_s$ and $v_g$ only.

(iii) Consider a scatterer $D$ with Dirichlet boundary conditions, and assume that there exists a function $g(\hat{\mathbf{r}}) \in L^2(S_1)$ such that $v_g(\mathbf{r}')$ as defined in (2) is

$$v_g(\mathbf{r}') = -\frac{e^{-ik|\mathbf{r}'|}}{k|\mathbf{r}'|} \ , \qquad \mathbf{r}' \in \partial D \ , \tag{4}$$

where $-k^2$ is not an eigenvalue of the interior Dirichlet problem. With this $v_g(\mathbf{r}')$, the expression for (3) simplifies to

$$\int_{S_1} f_\infty(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0, k) g^*(\hat{\mathbf{r}})ds(\hat{\mathbf{r}}) = \frac{1}{k}. \tag{5}$$

Assuming that the surface of the scatterer $\partial D$ is parametrised as

$$\partial D = \{\mathbf{r}' = h(\theta)\hat{\mathbf{r}}(\theta, \phi) \ ; 0 \leqslant \theta \leqslant \pi \ , \ 0 \leqslant \phi \leqslant 2\pi\} \ , \tag{6}$$

for some smooth, single-valued function $h(\theta)$, briefly explain how this expression, together with (4) and (5) , could be used in principle to solve the inverse problem of recovering the shape of the scatterer. *[Note that no analytical solution to this inverse problem is possible, and you are not asked to carry out any calculation when providing this brief explanation.]*

**2** A time-harmonic wave $\psi(\mathbf{r})e^{-i\omega t}$ in three-dimensional space is incident upon a inhomogeneity with refractive index $n(\mathbf{r})$ which occupies a volume $D$ in free space.

(i) Derive the first term of both the Born and the Rytov approximations for the space-dependent part of the total field at point $\mathbf{r}$, and denote them respectively by $\psi_B(\mathbf{r})$ and $\psi_R(\mathbf{r})$.

Show that the first order term in a power series expansion of the Rytov approximation is equal to the Born approximation

(ii) In the case where the incident field is a monochromatic plane wave propagating with wave number $k_0$ in a direction $\mathbf{r}_0$, derive far-field approximations to $\psi_B(\mathbf{r})$ and $\psi_R(\mathbf{r})$.

(iii) For the same incident wave as in (ii), assume that the refractive index in $D$ is $n(\mathbf{r}) = 1 + \mu W(\mathbf{r})$, where $\mu \ll 1$ and $W(\mathbf{r})$ is a statistically stationary random function of position with Gaussian p.d.f. with $\langle W \rangle = 0$, normalised so that $\langle W^2(\mathbf{r}) \rangle = 1$.

The mean intensity in the Rytov approximation is given by

$$I(\mathbf{r}) = \langle \psi_R^*(\mathbf{r}) \psi_R(\mathbf{r}) \rangle \tag{1}$$

Derive an expression for $I(\mathbf{r})$ in the far field in terms of the autocorrelation function of the 'scattering potential' $V = k_0^2[n^2(\mathbf{r}) - 1]$.

*[You may wish to use Re$(f) = \frac{1}{2}(f + f^*)$, for a complex function $f$; and the Taylor expansion when calculating $\langle \exp(\phi) \rangle$ for a random phase $\phi$.]*

**3** Consider the inverse problem

$$Ax = y \ , \tag{1}$$

where $x$ is the unknown and $A$ is a compact linear operator between two Hilbert spaces: $A : X \mapsto Y$.

(i) Give the definition of the Moore-Penrose generalised inverse solution $x^\dagger$, and of the Moore-Penrose operator $A^\dagger$ for (1).

Write the generalised inverse $x^\dagger$ in terms of the singular value system for $A$, $\{\sigma_n; v_n; u_n\}$ ($n \in \mathbb{N}$), hence explain why the Moore-Penrose operator can be unbounded.

(ii) Define what is meant by a regularisation strategy $R_\alpha$ for $A^\dagger$, and formally write the regularised solution $x_\alpha = R_\alpha y$ in terms of a filter function $f_\alpha(\sigma_i)$.

(iii) Derive the Landweber iteration scheme for solving (1), and explain

(a) why Landweber iteration can be a regularisation method and how the regularisation parameter $\alpha$ needs to be chosen;

(b) how the error in the approximation $x_n$ can be split into a component due to the regularisation and one due to the error in the data.

In your answer to (iii)(b), use the result that, given exact data $y$ and noisy data $y^{(\delta)}$ such that $\| y^{(\delta)} - y \| \leqslant \delta$, with $x_n$ and $x_n^{(\delta)}$ the corresponding Landweber iteration sequences, we have $\| x_n - x^{(\delta)} \| \leqslant \sqrt{n}\delta$.

(iv) Given that the $n^{th}$ Landweber iterate has the closed form representation

$$x_n = \sum_{k=0}^{n-1} (\mathbb{I} - A^* A)^k A^* y \ , \tag{2}$$

derive an expression for the filter function $f_\alpha(\sigma_i)$ for the Landweber iteration as a regularisation strategy.

*[You may wish to use the partial sum* $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$. *]*

## END OF PAPER