UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2019 9:00 am to 11:00 am

PAPER 334

ACTIVE BIOLOGICAL FLUIDS

Attempt ***ALL*** *questions.*

There are ***TWO*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

An infinite two-dimensional waving sheet is swimming under a semi-infinite viscous fluid using a transverse travelling-wave. At small amplitude in the waving motion, the dimensionless location of material points (x_s, y_s) in the frame swimming with the sheet is given by

$$x_s = x, \quad y_s = \epsilon \sin(x - t),$$

where (x, y) denotes the swimming frame of reference, t is the dimensionless time and $\epsilon \ll 1$. As a result of its waving motion, the sheet swims with steady speed $-U\mathbf{e}_x$ with respect to the fluid at rest at infinity. Inertial effects in the fluid, which occupies the region above the sheet, are neglected.

(a) What are the equations and boundary conditions satisfied by the streamfunction ψ in the frame moving with the sheet?

(b) Solving the problem as a perturbation expansion in powers of ϵ, i.e. $\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + ...$, derive the equation and the boundary conditions satisfied by ψ_1. Find the solution for ψ_1.

(c) Derive the equation and boundary conditions for ψ_2 and the general form of its solution. Show how this can be used to calculate the swimming speed at order ϵ^2, U_2, without requiring the full solution for ψ_2.

The fluid is now bounded from above by a flat no-slip wall located at the dimensionless position $y = d$.

(d) What are the boundary conditions satisfied by the streamfunction ψ on the wall in the swimming frame?

(e) Find the new solution for ψ_1 valid in the bounded fluid, $y_s \leqslant y \leqslant d$.

(f) Similarly to (c), solve for the swimming speed at order ϵ^2 in the presence of the wall, $\bar{U}_2$.

(g) Comparing the value of $\bar{U}_2$ to U_2, show that the presence of the wall always increases the value of the swimming speed.

Hint: You may use without proving it that the general unit-speed 2π-periodic travelling-wave solution to $\nabla^4 f(x, y, t) = 0$ is

$$f(x,y,t) = A + By + Cy^2 + Dy^3 + \mathcal{R}\left\{\sum_n \left[E_n e^{-ny} + F_n e^{ny} + y(G_n e^{-ny} + H_n e^{ny})\right] e^{in(x-t)}\right\}$$

2 An inextensible elastic slender filament is actuated periodically passively in a viscous fluid at low Reynolds number. At time t, the centreline of the filament is located at $\mathbf{r}(s,t)$ where s is the arclength along the filament, $0 \leqslant s \leqslant L$. The bending modulus for the filament is denoted by B.

(a) The elastohydrodynamic dynamic balance for the filament is

$$-\left[c_\perp \mathbf{1} + (c_\parallel - c_\perp)\mathbf{tt}\right] \cdot \frac{\partial \mathbf{r}}{\partial t} - B\frac{\partial^4 \mathbf{r}}{\partial s^4} + \frac{\partial}{\partial s}\left(T\frac{\partial \mathbf{r}}{\partial s}\right) = \mathbf{0}. \qquad (\dagger)$$

Explain briefly the origin of each of the three terms in this equations. What are the dimensions of $c_\perp$, $c_\parallel$, B and T?

(b) Denoting the shape of the filament by $y(x,t)$, $0 \leqslant x \leqslant L$, in Cartesian coordinates, show that for small amplitudes, (†) linearises to

$$c_\perp \frac{\partial y}{\partial t} = -B\frac{\partial^4 y}{\partial x^4} \cdot \qquad (\dagger\dagger)$$

Show that the total propulsive force exerted by the fluid on the filament along x is given by

$$\mathcal{F} = (c_\perp - c_\parallel)\int_0^L \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\mathrm{d}x.$$

Show that the value of $\mathcal{F}$ depends only on the dynamics at the boundary points ($x = 0,\ L$).

(c) Using dimensional arguments for the periodic motion of a filament in (††) with frequency ω, define the dimensionless *Sperm* number, Sp, as the ratio of two length scales, $\mathrm{Sp} = L/\ell_\omega$. Give an equation for ℓ_ω and verify its dimensions. Using ω^{-1} and ℓ_ω to non-dimensionalise times and lengths, show that the dimensionless version of (††) is

$$\frac{\partial y}{\partial t} = -\frac{\partial^4 y}{\partial x^4} \cdot \qquad (\dagger\dagger\dagger)$$

(d) The filament described by († † †) is oscillating periodically at $x = 0$ with dimensionless boundary conditions

$$y(x=0,t) = y_0 \cos t, \qquad \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0,t} = 0.$$

The filament is long (i.e. $L \gg \ell_\omega$) and the dimensionless boundary condition on its other end is given by $y(x \to \infty, t) = 0$.

Determine the solution for $y(x,t)$. Show that it is the sum of two exponentially damped waves travelling in opposite directions for which you will calculate the dimensionless wavespeeds.

Hint: it will be useful to define $C = \cos(\pi/8)$ snd $S = \sin(\pi/8)$.

END OF PAPER