UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2019 1:30 pm to 4:30 pm

PAPER 333

FLUID DYNAMICS OF CLIMATE

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Cartesian co-ordinates (x, y, z) are used with z denoting the upward vertical. The corresponding velocity components are (u, v, w). Unless stated otherwise, g is the gravitational acceleration.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The rotating shallow water equations can be written

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -g\nabla\eta + \mathbf{D} + \mathbf{W}, \tag{1}$$

$$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{u}H) = 0, \tag{2}$$

where $\mathbf{u} = (u, v)$ and $\nabla = (\partial_x, \partial_y)$ are the horizontal velocity and horizontal gradient operators, f is the local Coriolis parameter, $\hat{\mathbf{k}}$ is the vertical unit vector, g is the graviational acceleration, $H = H_0(x, y) + \eta(x, y, t)$ is the total fluid depth where H_0 is the resting depth and η is the dynamic sea surface height, and $\mathbf{D}$ and $\mathbf{W}$ represent dissipation and the wind stress, respectively. Starting from the above equations, derive an equation for flow in *Sverdrup balance*, clearly stating all assumptions made.

Consider a closed contour, C, in the interior of an ocean basin where the flow satisfies (1) and assume that H is constant. Show that the circulation around this contour satisfies

$$\oint_C \frac{\partial \mathbf{u}}{\partial t} \cdot \widehat{\mathbf{t}} ds = -\oint_C \Big[(\zeta + f)\widehat{\mathbf{k}} \times \mathbf{u}\Big] \cdot \widehat{\mathbf{t}} ds + \oint_C \mathbf{W} \cdot \widehat{\mathbf{t}} ds + \oint_C \mathbf{D} \cdot \widehat{\mathbf{t}} ds, \tag{3}$$

where $\zeta = (\nabla \times \mathbf{u}) \cdot \widehat{\mathbf{k}}$ is the vertical component of the relative vorticity and $\widehat{\mathbf{t}}$ is a unit vector tangential to the contour.

Consider an island in the middle of an ocean basin. Assuming there is no normal flow across the boundary of the island, show that the circulation around the island satisfies

$$\frac{\partial}{\partial t} \oint_{C_I} \mathbf{u} \cdot \widehat{\mathbf{t}} ds = \oint_{C_I} \mathbf{W} \cdot \widehat{\mathbf{t}} ds + \oint_{C_I} \mathbf{D} \cdot \widehat{\mathbf{t}} ds, \tag{4}$$

where C_I is the contour coincident with the boundary of the island.

Now, consider an island within an enclosed ocean basin where the flow is in Sverdrup balance. Assume that the dissipation $\mathbf{D}$ is non-zero on the eastern boundary of the island and on the western boundary of the basin, but can be neglected elsewhere. Further assume that there is no flow normal to the boundary of the ocean basin and without loss of generality, let the streamfunction be zero at the boundary of the ocean basin. Obtain an expression for the streamfunction ψ_I that encircles the island. By selecting a contour that avoids any regions with non-zero dissipation, show that ψ_I can be written in terms of $\mathbf{W}$.

Consider a square island in a rectangular ocean basin. Let the sides of the island have a length R, while the centre of the island, denoted $x = 0$, $y = 0$, is a distance S from the eastern boundary of the ocean basin. If the wind stress is $\mathbf{W}(y) = sin(\pi y)\widehat{\mathbf{x}}$ and the flow in the ocean basin is in Sverdrup balance, obtain an expression for the volume transport between the island and the eastern side of the ocean basin. Discuss the influence of the island on the strength of any boundary current that develops along the western boundary of the ocean basin.

2

The stratified quasi-geostrophic (QG) equations can be written

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0,$$

where

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N_0^2}\frac{\partial\psi}{\partial z}\right)$$

is the QG potential vorticity, $\nabla = (\partial_x, \partial_y)$ is the horizontal gradient operator, f_0 is the constant Coriolis parameter, N_0 is the constant buoyancy frequency, and $J(A, B) = A_x B_y - A_y B_x$ is the Jacobian operator.

Consider a semi-infinite fluid with buoyancy $b = N_0^2 z - \Lambda f_0 y$ bounded below by a flat, rigid surface located at $z = 0$ and Λ is constant. The velocity associated with the basic state, U, is in thermal wind balance and satisfies $U = 0$ at $z = 0$. Find the dispersion relation for quasi-geostrophic perturbations to this basic state.

Now, consider an unbounded fluid consisting of two regions each with constant buoyancy gradients such that

$$\begin{aligned} b &= N_1^2 z - \Lambda_1 f_0 y, \quad \text{for} \quad z < h(y), \\ b &= N_2^2 z - \Lambda_2 f_0 y, \quad \text{for} \quad z > h(y). \end{aligned}$$

In each region there is a flow in thermal wind balance and departures to this flow can be described in terms of the stratified QG equations. State and justify appropriate boundary conditions at $z = h$. Show that, consistent with the QG approximation, the boundary condition can be applied at a constant height. Find the phase speed of quasi-geostrophic perturbations with respect to the thermal wind. Show that in a suitable limit of N_1/N_2 the phase speed matches the phase speed for a semi-infinite fluid that you found above.

3

The effect of compressibility may be included in the β-plane primitive equations by re-defining the z coordinate in terms of pressure p as $z = -H\log(p/p_0)$, where H is a constant 'scale height' (equal to about 7 km for the Earth's atmosphere) and p_0 is a constant pressure. The resulting form of the equations is:

$$u_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})u - (f_0+\beta y)v = -\tilde{\Phi}_x, \tag{1}$$

$$v_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})v + (f_0+\beta y)u = -\tilde{\Phi}_y, \tag{2}$$

$$\tilde{\Phi}_z = \frac{R\tilde{T}}{H}, \tag{3}$$

$$\tilde{T}_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\tilde{T} + S(z)w = 0, \tag{4}$$

$$u_x + v_y + e^{z/H}(e^{-z/H}w)_z = 0, \tag{5}$$

where $S(z) > 0$ and $R > 0$ is a constant. (Note that the structure of these equations is very similar to those of the Boussinesq primitive equations considered in lectures.)

The velocity $\boldsymbol{u}$ may be divided into geostrophic and ageostrophic parts, respectively $\boldsymbol{u}_g$, where $\boldsymbol{u}_g = f_0^{-1}(-\tilde{\Phi}_y, \tilde{\Phi}_x, 0)$, and $\boldsymbol{u}_a = (u_a, v_a, w_a)$.

Starting from (1)-(5) derive the corresponding form of the quasi-geostrophic potential vorticity equation:

$$\{\frac{\partial}{\partial t} + \boldsymbol{u}_g\cdot\boldsymbol{\nabla}\}\{\psi_{xx} + \psi_{yy} + e^{z/H}\frac{\partial}{\partial z}(e^{-z/H}\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z})\} + \beta\psi_x = 0, \tag{6}$$

where $\psi = \tilde{\Phi}/f_0$. Give the definition of N in terms of other quantities appearing in (1)-(5).

In your derivation assume that typical horizontal and vertical length scales are respectively L and D, a typical velocity is U and a typical time scale is L/U and explain carefully why each of the three dimensionless quantities U/f_0L, $\beta L/f_0$ and $(f_0^2L^2H/RSD\min\{D,H\})(U/f_0L)$ must be small for (6) to provide a good description of the dynamics. [You may find it useful to consider the relative sizes of the components of $\boldsymbol{u}_a$, as implied by (5).]

In the remainder of the question assume that N is constant. Derive from (6) an equation describing small disturbances about a state of rest and show that there are plane-wave solutions of the form $\psi = \mathrm{Re}(\hat{\psi}_c\, e^{z/2H}e^{ikx+imz-i\omega t})$, with $\hat{\psi}_c$ constant, providing that ω is a certain function (to be given) of k and m. [You may assume here and subsequently that $k > 0$.]

Consider motion in the region $0 < z < \infty$ which is disturbed from a state of rest by some physical effect which sets $\psi = \psi_b(x,t) = \mathrm{Re}(\Psi_b\exp(ikx - i\omega t))$ on $z = 0$. (Continue to assume that the disturbances are small.) Seek a corresponding solution $\psi(x,y,z,t) = \mathrm{Re}(e^{z/2H}\hat{\chi}(z)\exp(ikx - i\omega t))$. Derive the equation satisfied by $\hat{\chi}(z)$ and solve it, distinguishing between different ranges of ω and explaining carefully what criteria you use to determine a unique solution. Deduce, for given k, the range of values of ω for which there are vertically propagating waves.

[Assume that it is appropriate to require that solutions have $|\hat{\chi}(z)|$ bounded as $z\to\infty$.]

4

The shallow-water equations on an equatorial β-plane, linearised about a state of rest with layer depth H, are

$$u_t - \beta y v = -g\eta_x, \tag{1}$$

$$v_t + \beta y u = -g\eta_y \tag{2}$$

$$\eta_t + H(u_x + v_y) = 0. \tag{3}$$

The gravity-wave speed c is equal to $\sqrt{gH}$.

Consider equatorially confined solutions of the form $[u, v, \eta] = \mathrm{Re}([\hat{u}(y), \hat{v}(y), \hat{\eta}(y)]e^{i(kx-\omega t)})$, with the constants k and ω representing, respectively, x-wavenumber and frequency. (Assume $\omega > 0$.)

Derive equations for $\hat{u}(y)$ and $\hat{\eta}(y)$ in terms of $\hat{v}(y)$.

Deduce that if $\omega = \pm kc$ there is a possible solution with $\hat{v}(y) = 0$, with $\hat{u}(y) \neq 0$ and $\hat{\eta}(y) \neq 0$. Find an ordinary differential equation describing the structure of $\hat{u}(y)$ in this case and deduce that only $\omega = kc$ gives an equatorially confined solution. Give the form of $\hat{u}(y)$ and $\hat{\eta}(y)$ in this case.

If $\hat{v}(y) \neq 0$ show that it satisfies the second order differential equation

$$\frac{d^2\hat{v}}{dy^2} + (\frac{\omega^2}{c^2} - \frac{\beta^2 y^2}{c^2} - k^2 - \frac{\beta k}{\omega})\hat{v} = 0.$$

The differential equation $V''(Y) - Y^2V(Y) = \lambda V(Y)$ has solutions which tend to zero as $|Y| \to \infty$ only when $\lambda = -(2n+1)$, $(n = 0, 1, 2, \dots)$ when $V(Y) = \tilde{D}_n(Y)$, with $D_0(Y) = \exp(-\frac{1}{2}Y^2)$.

Deduce that

$$\omega^2 - c^2k^2 - \beta k c^2/\omega = (2n+1)\beta c \qquad \text{for } n = 0, 1, 2, \dots. \tag{4}$$

What is the corresponding form of $\hat{v}(y)$?

Considering (4) as an equation that specifies k given ω show that for $n = 0$ there is a root $k = \omega/c - \beta/\omega$. (There is another root $k = -\omega/c$ which further analysis shows does not correspond to acceptable solutions for $\hat{u}$ and $\hat{\eta}$.) Write down the form of $\hat{v}(y)$ for $n = 0$, assuming that $\hat{v}(0) = 1$, and then deduce the corresponding expressions for $\hat{u}(y)$ and $\hat{\eta}(y)$. Analyse and comment on the dominant balance in the x-momentum equation (1) in the low-frequency limit $\omega \ll (\beta c)^{1/2}$ and in the high frequency limit $\omega \gg (\beta c)^{1/2}$.

Show that for $n \geqslant 1$ there are real solutions for k only if $\omega^2 \geqslant \beta c(n + \frac{1}{2} + \sqrt{n(n+1)})$ or $\omega^2 \leqslant \beta c(n + \frac{1}{2} - \sqrt{n(n+1)})$.

Using results from previous parts of the question, consider the response of the system to an equatorially confined forcing, localised in x, with specified frequency $\omega = \frac{1}{2}(\beta c)^{1/2}$. Will this forcing excite propagating waves and, if so, what will be the wavenumbers and will the waves be detected to the east or to the west of the forcing?

END OF PAPER