MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2019 9:00 am to 12:00 pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A uniform rain, of strength R m/s, falls on a two-dimensional aquifer of total length $x = L$, bounded below by an impermeable horizon along $z = 0$, and feeding a river located at $(x, z) = (0, 0)$. The porosity ϕ of the aquifer is uniform, but the permeability is depth-dependent, and given by $k = k_0(1 + \beta z)$, where k_0 is the reference permeability at $z = 0$ and $\beta > 0$. Derive an equation for the evolution of the height of the groundwater table, $h(x, t)$, subject to the conditions that the height at the river $h(0, t) = 0$ and that the total flux at the drainage divide $q(L, t) = 0$.

(a) Using your equation for the evolution of the groundwater table, first examine the long-time steady state that is reached if the rain persists and show that in this limit the depth of the groundwater table is given implicitly by

$$h^2 + \tfrac{1}{3}\beta h^3 = \frac{R}{u_b}(2Lx - x^2), \tag{1}$$

where $u_b = k_0 g\rho/\mu$ is the buoyancy velocity, g is the gravitational acceleration, ρ the density of water and μ the viscosity of water.

(b) Now examine the transient filling of the aquifer, assuming that the rainfall starts at $t = 0$ and that the aquifer is initially empty, $h(x, 0) = 0$. Using scaling arguments, show that the filling of the aquifer is self-similar, with different time dependence for early times $t \ll 2\phi/(R\beta)$ and intermediate times $2\phi/(R\beta) \ll t \ll \phi(L/R)^{2/3}(u_b\beta/2)^{-1/3}$, and derive the equations governing these self-similar solutions. Find an expression for the flux into the river in both early and intermediate time limits, and explain the origins of the temporal regimes.

(c) Finally, when the rain stops, $R = 0$ the aquifer discharges the remaining water into the river, again in two distinct regimes. Find two distinct self-similar regimes of discharge, describing the equations governing the shape of the self-similar profiles, the scalings for the discharge flux to the river in each regime, and the timescale for transition between regimes.

2

(a) Consider a rigid mushy region of pure ice crystals of density ρ_s and interstitial brine (salt solution) of density ρ_l, where $\rho_s \neq \rho_l$ but both densities are assumed to be independent of temperature and concentration.

Use mass conservation over a small representative region of mush to derive the equation

$$\nabla \cdot \mathbf{u} = (1-r)\frac{\partial \varphi}{\partial t},$$

where φ is the volume fraction of ice crystals, $r = \rho_s/\rho_l$, $\mathbf{u}$ is the Darcy velocity (flux per unit volume) of liquid relative to the ice crystals, and t is time.

If it is assumed that the heat capacity per unit volume and the thermal conductivity are independent of phase and that diffusivity of salt is negligible then the equations for heat and mass conservation and local thermodynamic equilibrium in the mushy layer are

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + \frac{L}{c_p}\frac{\partial \varphi}{\partial t},$$

$$(1-\varphi)\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C = rC\frac{\partial \varphi}{\partial t},$$

$$T - T_0 = -m(C - C_0),$$

where T is temperature, C is salt concentration of the liquid, κ is the thermal diffusivity, L is the latent heat of fusion, c_p is the specific heat capacity of ice, m is the constant slope of the liquidus, and C_0 is a reference concentration at which the liquidus temperature is T_0.

(b) Consider a system pulled vertically downwards at speed V through fixed heat exchangers such that, in a frame of reference fixed with the heat exchangers, the system is steady and the vertical position $z = 0$ corresponds to the eutectic temperature T_E. Far above the eutectic front, the composition and temperature of the liquid are C_0 and $T_\infty > T_0$ respectively.

(i) Assuming that the composite, eutectic solid also has density ρ_s, determine the velocity field throughout the system in terms of φ.

(ii) Determine the solid fraction φ in terms of the dimensionless temperature $\theta = (T - T_0)/\Delta T$. You may assume without proof that $\varphi = 0$ and $C = C_0$ at the mush–liquid interface.

In the following, assume that $\mathcal{S} \gg 1$ and $\mathcal{C} \gg 1$ with $\mathcal{S}/\mathcal{C} = \mathrm{O}(1)$, where $\mathcal{S} = L/c_{ps}\Delta T$, $\mathcal{C} = mC_0/\Delta T$ and $\Delta T = T_0 - T_E$.

(iii) Determine the dimensionless temperature field $\theta(z)$ in both the liquid and mushy regions.

(iv) Show that the thickness of the mushy layer h is given by

$$\frac{Vh}{\kappa} = \frac{1}{r\Omega}\ln\left(1 + \frac{\Omega}{\theta_\infty}\right),$$

where $\Omega = 1 + \mathcal{S}/r\mathcal{C}$ and $\theta_\infty = (T_\infty - T_0)/\Delta T$.

3

Consider a porous matrix composed of a mixture of solid rock and ice, saturated in water, all initially at the melting temperature T_m. Hot water, at temperature $T_m + \Delta T$, is injected into the matrix with volumetric flux U, causing the ice to melt, resulting in an increase in the liquid fraction, $\phi \to \phi + \Delta\phi$, and an increase in the permeability, $k \to k + \Delta k$, of the matrix. Assume that the rock is highly thermally conductive and so is at the same temperature as the water flowing through it.

(a) Using expressions of conservation of mass and energy at the evolving interface, derive the fluid velocity in the pre-existing matrix, and show that the melting front travels at a speed

$$V = \frac{U}{1 + \mathcal{S}\Delta\phi}, \tag{1}$$

where $\mathcal{S} = L/c_p\Delta T$ is the Stefan number and c_p is the specific heat capacity (of all phases). You should neglect any thermal diffusion, and instead consider only the jump in quantities across the evolving interface.

(b) Now consider the stability of the interface in the limit where $\mathcal{S}\Delta\phi \sim \mathcal{O}(1)$ and show that perturbations to the interface have growth rate

$$\sigma = \frac{\alpha U}{1 + \mathcal{S}\Delta\phi} \frac{\Delta k}{2k + \Delta k}, \tag{2}$$

where α is the transverse wavenumber of the instability.

(c) Interpret your results physically, suggesting potential mechanisms that could be included to provide a means for selecting the most unstable wavelength.

4

An axisymmetric, conical, peaked mountain has height H_m and sides of fixed angle to the horizontal α. The net annual accumulation of snow on the mountain is

$$a(x) = A\frac{H(x) - H_0}{H_m - H_0}, \qquad (1)$$

where A is a constant amplitude, $H(x)$ is the local elevation, x is distance along slope from the apex of the mountain, and the snowline is at a constant height $H_0 < H_m$.

(a) Starting from the lubrication forms of the Navier-Stokes and mass-conservation equations, show that the thickness of ice $h(x,t)$ satisfies the equation

$$\frac{\partial h}{\partial t} = -\frac{1}{x}\frac{\partial}{\partial x}\left(\frac{g\sin\alpha}{3\nu}xh^3\right) + A\frac{H(x) - H_0}{H_m - H_0},$$

where ν is the kinematic viscosity of ice and g is the acceleration due to gravity. You should state clearly any assumptions you make and any boundary conditions you apply. You may assume that ice flows as a Newtonian, viscous fluid, that $h \ll H_0$ and that $|\partial h/\partial x| \ll \tan\alpha$.

Determine the steady shape of the ice sheet, stating carefully any boundary conditions that you use and giving an explicit expression for the extent of the current x_N. Show that the volume of ice in the current is

$$V_0 = \lambda\left(\frac{\nu A \Delta H^7}{g\sin^8\alpha}\right)^{1/3}\cos\alpha,$$

where $\Delta H = H_m - H_0$, giving an expression for the coefficient λ as a dimensionless integral, which need not be evaluated.

(b) Some time after the steady state has been established, the net accumulation rate is set to zero (no further accumulation nor loss of ice). Find a similarity solution for the subsequent flow of the ice, determining its self-similar profile, its extent $x_N(t)$ and its thickness at its terminus $h_N = h(x_N)$.

(c) Imagine that the mountain is initially bare of ice $h(x,t) \equiv 0$ and that the ice accumulation (1) is suddenly switched on at time $t = 0$ and maintained thereafter. Use scaling analysis to describe qualitatively the subsequent evolution of $h(x,t)$, identifying the time scales of any significant transitions in behaviour.

END OF PAPER