UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 9:00 am to 12:00 pm

PAPER 331

HYDRODYNAMIC STABILITY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider infinitesimal, two-dimensional perturbations about a parallel shear flow in an inviscid, stratified fluid in a finite depth domain $z \in [-L, L]$ between impermeable boundaries:

$$\begin{aligned}
\mathbf{u} &= \overline{U}(z)\hat{\mathbf{x}} + \mathbf{u}'(x,z,t); U_{\min} \leqslant \overline{U} \leqslant U_{\max}, \\
p &= \overline{p}(z) + p'(x,z,t), \\
\rho &= \overline{\rho}(z) + \rho'(x,z,t), \\
\left[\mathbf{u}', p', \rho'\right] &= \left[\hat{\mathbf{u}}(z), \hat{p}(z), \hat{\rho}(z)\right] \exp[ik(x-ct)]; \hat{\mathbf{u}} = (\hat{u}, \hat{w}),
\end{aligned}$$

where the wavenumber k is assumed real, the phase speed $c = c_r + ic_i$ may in general be complex, and gravity $\mathbf{g} = -g\hat{\mathbf{z}}$ acts in the negative $z-$direction.

(a) Applying the Boussinesq approximation appropriately, show that the vertical velocity eigenfunction $\hat{w}$ satisfies the Taylor-Goldstein equation:

$$\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} - \frac{\hat{w}}{(\overline{U}-c)}\frac{d^2}{dz^2}\overline{U} + \frac{N^2\hat{w}}{(\overline{U}-c)^2} = 0; \; N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz},$$

where N is the buoyancy frequency and ρ_0 is an appropriate reference density.

(b) Let $\hat{w} = (\overline{U} - c)^a q$ for arbitrary real a.

(i) Show that

$$\begin{aligned}
\int_{-L}^{L} (\overline{U}-c)^{2a}\left[|q_z|^2 + k^2|q|^2\right] dz &= \int_{-L}^{L}\left[\left(N^2 + a(a-1)[\overline{U}_z]^2\right)(\overline{U}-c)^{2a-2}\right]|q|^2 dz \\
&\quad + \int_{-L}^{L}\left[(a-1)\overline{U}_{zz}(\overline{U}-c)^{2a-1}\right]|q|^2 dz,
\end{aligned}$$

where subscripts denote differentiation with respect to z.

(ii) Using an appropriate choice of a, show that the flow must be marginally stable if

$$4N^2 > \left(\frac{d\overline{U}}{dz}\right)^2,$$

for all $z \in [-L, L]$.

(iii) Using a different choice of a, show that $U_{\min} < c_r < U_{\max}$ if the flow is unstable.

2

Consider infinitesimal, two-dimensional perturbations about a parallel shear flow in an inviscid, stratified fluid:

$$\begin{aligned}
\mathbf{u} &= \overline{U}(z)\hat{\mathbf{x}} + \mathbf{u}'(x,z,t),\\
p &= \overline{p}(z) + p'(x,z,t),\\
\rho &= \overline{\rho}(z) + \rho'(x,z,t),\\
[\mathbf{u}', p', \rho'] &= [\hat{\mathbf{u}}(z), \hat{p}(z), \hat{\rho}(z)]\exp[ik(x-ct)];\ \hat{\mathbf{u}} = (\hat{u}, \hat{w}),
\end{aligned}$$

where the wavenumber k is assumed real, the phase speed $c = c_r + ic_i$ may in general be complex, and gravity $\mathbf{g} = -g\hat{\mathbf{z}}$ acts in the negative $z-$direction.

(a) Assume that there is a piecewise constant distribution of background density $\overline{\rho}$. Also assume that there is either a piecewise constant distribution or a piecewise linear distribution of background velocity $\overline{U}$. Show that the appropriate jump conditions at interfaces, where at least one of the density, vorticity or velocity are discontinuous, are given by:

$$\left[\frac{\hat{w}}{(\overline{U}-c)}\right]_-^+ = 0; \quad \left[(\overline{U}-c)\frac{d}{dz}\hat{w} - \hat{w}\frac{d}{dz}\overline{U} - \frac{g\overline{\rho}}{\rho_0}\left(\frac{\hat{w}}{(\overline{U}-c)}\right)\right]_-^+ = 0.$$

In the Boussinesq approximation, you are given that the vertical velocity eigenfunction $\hat{w}$ satisfies the Taylor-Goldstein equation:

$$\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} - \frac{\hat{w}}{(\overline{U}-c)}\frac{d^2}{dz^2}\overline{U} + \frac{N^2\hat{w}}{(\overline{U}-c)^2} = 0;\ N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz},$$

where N is the buoyancy frequency and ρ_0 is an appropriate reference density.

(b) Consider a three-layer flow:

$$\overline{U} = \begin{cases} \frac{\Delta U z}{h} \\ \frac{\Delta U z}{h} \\ \frac{\Delta U z}{h} \end{cases}, \qquad \overline{\rho} = \begin{cases} \rho_0 - \frac{\Delta\rho}{2} & z > \frac{h}{2}; \\ \rho_0 & |z| < \frac{h}{2}; \\ \rho_0 + \frac{\Delta\rho}{2} & z < -\frac{h}{2}. \end{cases}$$

(i) Show that $\tilde{c} = 2c/\Delta U$ satisfies

$$\tilde{c}^4 - \left(2 + \frac{J}{\alpha}\right)\tilde{c}^2 + \frac{(2\alpha - J)^2 - J^2 e^{-4\alpha}}{4\alpha^2} = 0,$$

where $\alpha = kh/2$ and $J = g\Delta\rho h/[\rho_0 \Delta U^2]$.

(ii) Hence show that the flow is unstable for

$$\frac{2\alpha}{1+e^{-2\alpha}} < J < \frac{2\alpha}{1-e^{-2\alpha}}.$$

(iii) Interpret this instability in terms of a wave resonance in the limit of large wavenumber.

3

Consider the equation

$$\frac{d\boldsymbol{x}}{dt} = \mathbb{L}\boldsymbol{x} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (\star)$$

The energy $E(t)$ of the solution at a given time is defined as $E = x_1(t)^2 + x_2(t)^2$.

(a) Show by induction that

$$\mathbb{L}^n = \begin{bmatrix} \lambda_1^n & 0 \\ \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} & \lambda_2^n \end{bmatrix},$$

and hence that

$$\mathbb{A}(t) = e^{\mathbb{L}t} = \sum_{n=0}^{\infty} \frac{(\mathbb{L}t)^n}{n!} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} & e^{\lambda_2 t} \end{bmatrix}.$$

(b) Confirm that $\mathbb{A}(t)\boldsymbol{x}(0)$ is the solution to $(\star)$.

(c) Demonstrate that $\mathbb{L}$ is a non-normal matrix. Find its eigenvectors and demonstrate that they are not orthogonal.

(d) For the case $\lambda_1, \lambda_2 < 0$, find the set of initial conditions (x_1, x_2) where the energy grows immediately. Hence find conditions on λ_1 and λ_2 under which no energy growth is possible.

(e) Find a general expression for the maximum energy growth $G(t) = E(t)/E(0)$ at some time t. If $\lambda_2 > \lambda_1$, show that

$$\lim_{t \to \infty} G = (1 + \alpha^2) e^{2\lambda_2 t}, \qquad (\dagger)$$

where the constant α^2 is to be found. Find the initial condition to achieve the optimal long-time growth of $(\dagger)$ and comment on its relationship to the eigenvector of $\mathbb{L}$ corresponding to the eigenvalue λ_2.

4

Two-dimensional Rayleigh-Darcy convection in a porous layer is described by the equations

$$\nabla^2\psi = -R\frac{\partial\theta}{\partial x} \qquad \& \qquad \frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial z} = \nabla^2\theta$$

with boundary conditions $\psi = 0$ and $\theta = -1$ at the top boundary $z = 1$ and $\psi = 0$ and $\theta = 0$ at the lower boundary $z = 0$ where $\psi(x, z, t)$ is the streamfunction, $\theta(x, z, t)$ is the temperature and R is the Rayleigh number.

(a) Show that $\psi_0 = 0$ and $\theta_0 = -z$ is the basic conductive solution.

(b) Taking perturbations $\psi = \psi_0 + \psi'$ and $\theta = \theta_0 + \theta'$, linearize the system. Hence show that with normal modes of the form $(\psi', \theta')(x, z, t) = (\hat{\psi}, \hat{\theta})(z)e^{st+ikx}$ the eigenvalue s is given by

$$s = \frac{k^2R}{k^2 + n^2\pi^2} - (k^2 + n^2\pi^2) \qquad \text{for } n = 1, 2, \ldots.$$

Find the lowest value of $R = R_c$ for convection to occur and confirm that the most unstable mode has a wavelength of 2 in the x-direction.

(c) Now consider the weakly nonlinear saturation of the convection for $R = R_c + \varepsilon^2$ where $\varepsilon \ll 1$. By introducing the slow time scale T such that $T = \varepsilon^2 t$, show that the equations can be written as

$$\nabla^2\psi' + R_c\frac{\partial\theta'}{\partial x} = -\varepsilon^2\frac{\partial\theta'}{\partial x},$$

$$\nabla^2\theta' - \frac{\partial\psi'}{\partial x} = \varepsilon^2\frac{\partial\theta'}{\partial T} + \frac{\partial\psi'}{\partial z}\frac{\partial\theta'}{\partial x} - \frac{\partial\psi'}{\partial x}\frac{\partial\theta'}{\partial z}.$$

(d) By assuming expansions

$$\psi'(x, z, t) = \varepsilon A(T)\psi_1(x, z) + \varepsilon^2 A(T)^2\psi_2(x, z) + \varepsilon^3 A(T)^3\psi_3(x, z) + \ldots$$

$$\theta'(x, z, t) = \varepsilon A(T)\theta_1(x, z) + \varepsilon^2 A(T)^2\theta_2(x, z) + \varepsilon^3 A(T)^3\theta_3(x, z) + \ldots$$

find θ_1 if $\psi_1 = \cos\pi x \sin\pi z$. By considering the problem at $O(\varepsilon^2)$, find ψ_2 and θ_2.

(e) Write down the equations which ψ_3 and θ_3 must solve, and using the fact that

$$\int_0^1\int_0^2 \psi_1\left[\nabla^4\psi_3 + R_c\frac{\partial^2\psi_3}{\partial x^2}\right]dxdz = 0,$$

derive the amplitude equation

$$\frac{dA}{dT} = \alpha A - \beta A^3,$$

where the coefficients α and β need only be expressed in terms of integrals involving ψ_1, θ_1, ψ_2 and θ_2.

END OF PAPER