UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2019 9:00 am to 12:00 pm

PAPER 329

SLOW VISCOUS FLOW

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The concentration C of insoluble surfactant on the surface of an inviscid bubble immersed in a viscous fluid obeys the equation

$$\frac{DC}{Dt} = -C[\boldsymbol{\nabla}_s\cdot\mathbf{u}_s + (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\nabla}_s\cdot\mathbf{n}] + D_s\nabla_s^2 C\,, \qquad (\dagger)$$

where $\mathbf{n}$ is the unit normal out of the bubble, $\mathbf{u}_s = \mathbf{I}_s\cdot\mathbf{u}$ the tangential fluid velocity and $\boldsymbol{\nabla}_s = \mathbf{I}_s\cdot\boldsymbol{\nabla}$ the tangential gradient operator; $(\mathbf{I}_s)_{ij} = \delta_{ij} - n_i n_j$ is the local projection tensor onto the interface. Describe the physical interpretation of each of the terms in (†).

Consider the steady concentration $C(\mathbf{x})$ on a spherical bubble of radius a with an interfacial velocity $\mathbf{u} = \mathbf{I}_s(\mathbf{n})\cdot\mathbf{A}\cdot\mathbf{x}$, where $\mathbf{A}$ is a constant, symmetric, traceless second-rank tensor, and $\mathbf{x}$ is the position vector from the centre of the bubble. Under what condition on a, D_s and $|\mathbf{A}|$ is it possible to simplify (†) by writing $C = C_0 + C'$, where $|C'| \ll C_0$ and C_0 is uniform? Assuming that this condition is satisfied, show that

$$\boldsymbol{\nabla}_s\cdot\mathbf{u}_s = -3\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n} \quad \text{and} \quad C' = K\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}\,,$$

where the constant of proportionality K should be found.

[*You may use the results* $\boldsymbol{\nabla}_s\mathbf{n} = \mathbf{I}_s/a$ *and* $\nabla_s^2(n_i n_j) = 2(\delta_{ij} - 3n_i n_j)/a^2$.]

For $C' \ll C_0$ the surface-tension coefficient is given by $\gamma(C) = \gamma_0 - \gamma_1 C'$, where $\gamma_0 = \gamma(C_0)$ and γ_1 is a positive constant. Viscous stresses and the variation of surface tension deform the shape of the drop slightly to $r = a\left(1 + \dfrac{\mathbf{x}\cdot\mathbf{D}\cdot\mathbf{x}}{r^2}\right)$, with curvature

$$\kappa = \frac{2}{a} + \frac{4}{a}\frac{\mathbf{x}\cdot\mathbf{D}\cdot\mathbf{x}}{a^2} + O(|\mathbf{D}|^2)\,,$$

where $\mathbf{D}$ is a constant, symmetric, traceless second-rank tensor, and $|\mathbf{D}| \ll 1$. Write down the general stress boundary condition for a fluid–fluid interface with surface tension γ and curvature κ, and show that in this case it linearises to

$$[\boldsymbol{\sigma}\cdot\mathbf{n}]_-^+ = \frac{2\gamma_0\mathbf{n}}{a} + \frac{4\gamma_0}{a}(\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})\mathbf{n} + \frac{2K\gamma_1}{a}\big(\mathbf{I}_s\cdot\mathbf{A}\cdot\mathbf{n} - (\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})\mathbf{n}\big)\,.$$

Assuming that $\mathbf{u} \to \mathbf{E}\cdot\mathbf{x}$ as $r/a \to \infty$ and that $\mathbf{A} = \alpha\mathbf{E}$ and $\mathbf{D} = \beta\mathbf{E}$ in a steady state, explain why the Papkovich–Neuber potentials for the flow can be written in the form

$$\boldsymbol{\Phi} = \frac{Pa^3}{3}\mathbf{E}\cdot\boldsymbol{\nabla}\frac{1}{r} \qquad \chi = \frac{1}{2}\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x} + \frac{Qa^5}{3}\mathbf{E}:\boldsymbol{\nabla}\boldsymbol{\nabla}\frac{1}{r}\,,$$

where P and Q are constants. Given that these potentials correspond to

$$\mathbf{u} = (1 + P - 3Q)(\mathbf{n}\cdot\mathbf{E}\cdot\mathbf{n})\mathbf{x} + (1 + 2Q)\mathbf{I}_s\cdot\mathbf{E}\cdot\mathbf{x}$$

$$\boldsymbol{\sigma}\cdot\mathbf{n} = 2\mu(1 - 3P + 12Q)(\mathbf{n}\cdot\mathbf{E}\cdot\mathbf{n})\mathbf{n} + 2\mu(1 + P - 8Q)\mathbf{I}_s\cdot\mathbf{E}\cdot\mathbf{n}$$

on $r = a$, show that in steady state the deformation of the bubble is given by

$$\mathbf{D} = \frac{5\mu\mathbf{E}a}{\gamma_0}\left(\frac{2 + M}{5 + 2M}\right),$$

where $M = K\gamma_1/\mu a$.

Show that $\alpha \to 0$ as $M \to \infty$ and interpret this result physically.

2

Consider axisymmetric stretching of a thin planar sheet of viscous fluid that varies on a much longer radial length scale than its thickness. Let the sheet have thickness $h(r,t)$, where $|\partial h/\partial r| \ll 1$, and let the velocity have components $(u,0,w)$ with respect to cylindrical polar coordinates (r,θ,z).

By considering the rate of extension of small material line elements, or otherwise, write down the components e_{rr}, $e_{\theta\theta}$ and e_{zz} of the strain-rate tensor. Check that your answers are consistent with the incompressibility condition $(1/r)\partial(ru)/\partial r + \partial w/\partial z = 0$.

The top and bottom surfaces of the sheet are acted on by an applied external pressure $p_{ext}(r)$, but are free of any shear stress. Surface tension on these surfaces is negligible and there is no body force. Show that

$$\sigma_{\theta\theta} = -p_{ext} + 4\mu\frac{u}{r} + 2\mu\frac{\partial u}{\partial r}$$

and find a similar expression for σ_{rr}.

Sketch the forces acting on the four sides of a 'pineapple-chunk' portion of the sheet extending from r to $r+\delta r$ and from $-\delta\theta$ to $+\delta\theta$, and write down the r-component of the force acting on the top and bottom surfaces of the chunk. Hence derive the equation

$$2\mu\left[\frac{\partial}{\partial r}\Big(2rh\frac{\partial u}{\partial r} + hu\Big) - h\Big(\frac{2u}{r} + \frac{\partial u}{\partial r}\Big)\right] = rh\frac{\partial p_{ext}}{\partial r}\ .$$

Write down the equation for $\partial h/\partial t$ resulting from mass conservation in the chunk.

(a) The spread of a very viscous circular oil slick over a flat sea is described by the above equations with $p_{ext} = \Delta\rho\, gh$ arising from the jump in modified pressures. Assume that the boundary condition at the edge of the slick $r = R(t)$ is $h\sigma_{rr} = -\frac{1}{2}\Delta\rho\, gh^2$ (with $h(R) > 0$).

Find a similarity solution for the spread of a fixed volume V released at the origin. [*Hint*: Find the velocity similarity function $U(\eta)$ from mass conservation, show that $dH/d\eta = 0$ and determine the constants.]

(b) At $t = 0$ a small hole is made in the centre of a circular viscous sheet of uniform thickness h_0, which is supported by a stationary rigid boundary of radius R_0. The growth of the hole, of radius $R(t)$, due to surface tension acting on its edge is described by the above equations with $p_{ext} = 0$ (since curvatures are small away from the edge). By sketching a section through the edge of the hole, or otherwise, give a brief physical interpretation of the boundary condition $h\sigma_{rr} = -2\gamma$ at $r = R(t)$.

Show that if h is uniform at any time then u is of the form $Ar + B/r$ and thus h remains uniform. Find A and B, and show that $R(t)$ obeys the equation

$$\frac{dx}{dt} = \frac{\gamma}{\mu h_0}\frac{x(1-x^2)^2}{1+3x^2},$$

where $x(t) = R(t)/R_0$.

3

A rigid sphere of radius a moves with velocity $\mathbf{U} = (U, 0, 0)$ and angular velocity $\boldsymbol{\Omega} = (0, \Omega, 0)$ through a semi-infinite body of viscous fluid in $z > 0$ bounded by a rigid stationary plane at $z = 0$. Show that the force $\mathbf{F}$ and couple $\mathbf{G}$ exerted by the sphere on the fluid each have only one nonzero cartesian component, F and G say.

Let the centre of the sphere be at $(0, 0, (1+\epsilon)a)$ in the co-moving frame, with $0 < \epsilon \ll 1$. Use the equations of lubrication theory to derive the partial-differential equation (the Reynolds equation) that determines the pressure $p(x, y)$ in the thin gap beneath the sphere in terms of the gap thickness $h(x, y)$.

Verify that this equation is satisfied by $p = Ax/h^2$ for a suitable choice of A. Deduce that

$$\frac{\sigma_{xz}}{\mu} = \frac{6(U - \Omega a)x^2}{5ah^2} + \frac{2U + 8\Omega a}{5h} \quad \text{on } z = 0\,.$$

Obtain similar expressions for σ_{xz} on $z = h$ and for σ_{yz} on $z = 0$ and $z = h$.

The cartesian components of $\mathbf{G}$ can be calculated to leading order as integrals involving σ_{xz} and σ_{yz} over the thin-gap region $r \ll a$. Explain briefly why these expressions are consistent (at leading order) with two of these components being zero.

Show that the *leading-order* contribution to F from the thin-gap region $r \ll a$ is given by

$$F = \frac{4}{5}\pi\mu a(4U + \Omega a)\ln\frac{1}{\epsilon}\,.$$

[*You may assume that*

$$\int_0^{O(a)} \frac{r^{2n+1}\,dr}{h^n} = C_n \ln\frac{1}{\epsilon} + O(1)\,,$$

the leading-order logarithmic term comes from the region $L \ll r \ll a$ where $\epsilon a \ll h(r) \ll a$, and the prefactor C_n can be calculated by simple approximations for h and L.]

Similarly, calculate the leading-order contribution to G from the thin gap. What check does the reciprocal theorem provide on (part of) this calculation?

A uniform rigid sphere with buoyancy-adjusted weight F falls through viscous fluid very close to a vertical wall. Apply your previous results to determine the leading-order fall speed U and rotation rate Ω of the sphere.

END OF PAPER