UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS **Part III**

Friday, 31 May, 2019 9:00 am to 11:00 am

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the space of Schwartz functions $\mathcal{S}(\mathbf{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$, specifying the notion of convergence in each.

Define the Fourier transform on $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}'(\mathbf{R}^n)$. Show that for a tempered distribution $u \in \mathcal{S}'(\mathbf{R}^n)$ and α, β multi-indices

$$(D^\alpha u)\hat{} = \lambda^\alpha \hat{u}, \qquad (x^\beta u)\hat{} = (-1)^{|\beta|} D^\beta \hat{u}.$$

Consider $u \in \mathcal{S}'(\mathbf{R})$ defined by the function $u(x) = \frac{1}{2}\log\left(1 + x^2\right)$. By considering $u'(x)$, or otherwise, show that for all $\varphi \in \mathcal{S}(\mathbf{R})$

$$\langle \hat{u}, \varphi \rangle = c\varphi(0) - \pi \lim_{\epsilon \downarrow 0} \int_{|x|>\epsilon} \left(\frac{\varphi(x) - \varphi(0)}{|x|} \right) e^{-|x|}\, \mathrm{d}x$$

where c is a constant you should determine. Standard results from the course can be used if clearly stated. You might find it helpful to consider the family of Fourier transform pairs

$$\varphi_n(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4n}, \quad \hat{\varphi}_n(\lambda) = \sqrt{n} e^{-n\lambda^2}.$$

2

Define the space of distributions $\mathcal{D}'(\mathbf{R})$ and the corresponding space of test functions $\mathcal{D}(\mathbf{R})$, specifying the notion of convergence in each.

(a) Prove that if $v' = 0$ in $\mathcal{D}'(\mathbf{R})$ then $v = \text{const}$.

(b) Find the most general solution to $xv = 1$ in $\mathcal{D}'(\mathbf{R})$.

Hence find the most general solution $u \in \mathcal{D}'(\mathbf{R})$ to each of the equations

(i) $xu' + u = \delta_0'$,

(ii) $u'' = \delta_0' - 2\delta_1$,

(iii) $(x^2 - 1)u' = \delta_0$.

3

State and prove the Paley-Wiener-Schwartz theorem.

For $\varphi \in \mathcal{S}(\mathbf{R}^n)$ define the *dilation by* $t > 0$ by $\delta_t \varphi(x) = \varphi(tx)$. Using a duality argument, show that this definition extends to $u \in \mathcal{S}'(\mathbf{R}^n)$ via

$$\langle \delta_t u, \varphi \rangle = t^{-n} \left\langle u, \delta_{1/t} \varphi \right\rangle \quad \forall \varphi \in \mathcal{S}(\mathbf{R}^n).$$

Using the Paley-Wiener-Schwartz theorem, show that if $u \in \mathcal{S}'(\mathbf{R}^n)$ and $\operatorname{supp}(u) \subset \{x \in \mathbf{R}^n : |x| \leqslant 1\}$, then $\operatorname{supp}(\delta_t u) \subset \{x \in \mathbf{R}^n : |tx| \leqslant 1\}$.

END OF PAPER