MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2019 9:00 am to 11:00 am

PAPER 326

INVERSE PROBLEMS IN IMAGING

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Spectral regularisation

You may use results from the lectures provided these are clearly stated.

Let $A \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, where $\mathcal{U}, \mathcal{V}$ are Hilbert spaces.

a) Give the definition of a *regularisation* of $A^\dagger$. Give one example of a linear and one of a nonlinear regularisation. Give the definition of a *convergent regularisation.*

b) Prove the following statement. Let $\{R_\alpha\}_{\alpha>0}$ be a linear regularisation and $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ an a priori parameter choice rule. Suppose that $\lim_{\delta\to 0} \alpha(\delta) = 0$ and $\lim_{\delta \to 0} \delta \|R_{\alpha(\delta)}\|_{\mathcal{L}(\mathcal{V},\mathcal{U})} = 0$. Then (R_α, α) is a convergent regularisation.

c) Let $A \in \mathcal{K}(\mathcal{U}, \mathcal{V})$ be a compact operator with singular system $\{\sigma_j, x_j, y_j\}_{j\in\mathbb{N}}$. Asymptotic regularisation consists in choosing the regularised solution u_α as $u_\alpha = x\left(\frac{1}{\alpha}\right)$, where $x(t)$ solves the following initial value problem

$$\begin{cases} x'(t) = -A^*(Ax(t) - f), & t > 0, \\ x(0) = 0. \end{cases}$$

Derive the spectral representation of u_α in terms of the singular system of A.

Hint: Expand $x(t)$ in the basis of singular vectors of A. You may also use the fact that the solution of the following initial value problem

$$\begin{cases} y'(t) = ay(t) + b, & t > 0, \\ y(0) = 0, \end{cases}$$

where a, b are constants, is given by $y(t) = \frac{b}{a}(e^{at} - 1)$.

2 Variational regularisation

You may use results from the lectures provided these are clearly stated.

Let $\mathcal{U}, \mathcal{V}$ be Banach spaces.

a) Define the *weak* and the *weak-$*$ convergences* in a Banach space. What is the relationship between the weak and the weak-$*$ convergences in a reflexive Banach space?

b) What is the *Radon-Riesz property*? Prove that the norm in a Hilbert space has the Radon-Riesz property.

c) Consider an inverse problem $Au = f$, where $A \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, $\mathcal{U}, \mathcal{V}$ are Banach spaces and $f \in \mathcal{D}(A^\dagger)$, where $\mathcal{D}(A^\dagger)$ is the domain of the Moore-Penrose inverse $A^\dagger$. Let $f_\delta \in \mathcal{V}$ be such that $\|f - f_\delta\|_\mathcal{V} \leqslant \delta$ and $\mathcal{J} \colon \mathcal{U} \to \mathbb{R}$ be a regularisation functional. The following regularisation method is referred to as the residual method

$$u_\delta \in \operatorname*{arg\,min}_{u \in \mathcal{U} \colon \|Au - f_\delta\|_\mathcal{V} \leqslant \delta} \mathcal{J}(u).$$

Suppose that $\mathcal{J}(u) \geqslant 0$ for all u, $\mathcal{J}$ is strongly l.s.c. and its sublevel sets $\{u \in \mathcal{U} \colon \mathcal{J}(u) \leqslant C\}$ are strongly sequentially compact. Show that under these assumptions $u_\delta \to u^\dagger_\mathcal{J}$ as $\delta \to 0$, where $u^\dagger_\mathcal{J}$ is a $\mathcal{J}$-minimising solution.

Hint: You may assume that under these assumptions a $\mathcal{J}$-minimising solution exists and the optimisation problem above has a minimiser.

3 Total Variation

You may use results from the lectures provided these are clearly stated.

Let $\mathcal{U} = L^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded.

a) Give the definition of *Total Variation* and prove that it is proper and convex. Is Total Variation strictly convex? [Prove or give counter example.] Define the BV *space*. Is Total Variation coercive on BV? [Prove or give counter example.]

b) Give the definition of an *absolute one-homogeneous functional*. Show that if $\mathcal{J} : \mathcal{U} \to \mathbb{R}_{>0}$ is absolute one-homogeneous and convex and $p \in \partial\mathcal{J}(u)$ then $\mathcal{J}(u) = \langle p, u \rangle$.

c) Let $\Omega = [-1, 1]$ and $0 < R < 1$. Find the Total Variation of $u_R \colon \Omega \to \mathbb{R}$ defined as follows

$$u_R = \begin{cases} 1, & \text{if } |x| \leqslant R, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: You may use the fact that the hat function

$$\omega_\varepsilon(x) = \begin{cases} e^{-\frac{\varepsilon^2}{\varepsilon^2 - x^2}} & \text{if } |x| < \varepsilon, \\ 0 & \text{otherwise} \end{cases}$$

is in $C_0^\infty([-1, 1])$ *for any* $\varepsilon \leqslant 1$.

4 Source Conditions and Convergence Rates

You may use results from the lectures provided these are clearly stated.

Suppose that $\mathcal{U}$ is a Hilbert space.

a) Give the definition of the *generalised Bregman distance* $D^p_{\mathcal{J}}(u,v)$ associated with a convex functional $\mathcal{J}$ and a subgradient $p \in \partial\mathcal{J}(v)$. Show that if $\mathcal{J}$ is strictly convex, then $D^p_{\mathcal{J}}(u,v) = 0$ implies $u = v$. Show that in general $D^p_{\mathcal{J}}(u,v) = 0$ does not imply $u = v$. [Give counter example.]

b) State the *source condition* for a $\mathcal{J}$-minimising solution $u^\dagger_{\mathcal{J}}$. Show that any $u \in \mathcal{U}$ satisfying $Au = f$ and the source condition is a $\mathcal{J}$-minimising solution.

c) The following variational regularisation problem is called the exact penalisation model

$$\min_{u \in \mathcal{U}} \|Au - f\|_{\mathcal{V}} + \alpha\mathcal{J}(u),$$

where $f \in \mathcal{R}(A)$ is the exact data and $\mathcal{J} \colon \mathcal{U} \to \bar{\mathbb{R}}$ is proper, convex, l.s.c., and absolute one-homogeneous. Suppose that the source condition is satisfied at a $\mathcal{J}$-minimising solution $u^\dagger_{\mathcal{J}}$ with the source element $\mu^\dagger$ and $\alpha < \frac{1}{\|\mu^\dagger\|_{\mathcal{V}}}$. Show that in this case each minimisier u_α is a $\mathcal{J}$-minimising solution of $Au = f$ and

$$D^{p^\dagger}_{\mathcal{J}}(u_\alpha, u^\dagger_{\mathcal{J}}) = 0,$$

where $p^\dagger = A^*\mu^\dagger$.

END OF PAPER