UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 1:30 pm to 3:30 pm

PAPER 322

BINARY STARS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A binary star consists of two point-like components of masses M_1 and M_2 with star 2 at $\boldsymbol{r}(t)$ from star 1. Show that the orbital energy

$$E = \frac{1}{2}\mu|\dot{\boldsymbol{r}}|^2 - \frac{GM_1M_2}{r},$$

angular momentum

$$\boldsymbol{J} = \mu\boldsymbol{r} \times \dot{\boldsymbol{r}} = \mu\boldsymbol{h},$$

defining $\boldsymbol{h}$, where $\mu = M_1M_2/M$, with $M = M_1 + M_2$, and $r = |\boldsymbol{r}|$. Use Newton's law of gravity for both stars to demonstrate that both E and $\boldsymbol{J}$ and the Laplace–Runga–Lenz vector, defined by

$$\boldsymbol{e} = \frac{1}{GM}\left\{\dot{\boldsymbol{r}} \times \boldsymbol{h} - \frac{GM}{r}\boldsymbol{r}\right\},$$

are conserved.

Suppose there is an additional perturbing force $\boldsymbol{f}$ such that the acceleration

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r} + \boldsymbol{f}.$$

Show that, instantaneously,

$$\dot{E} = \mu\dot{\boldsymbol{r}} \cdot \boldsymbol{f},$$
$$\dot{\boldsymbol{h}} = \boldsymbol{r} \times \boldsymbol{f}$$

and

$$GM\dot{\boldsymbol{e}} = \boldsymbol{f} \times \boldsymbol{h} + \dot{\boldsymbol{r}} \times (\boldsymbol{r} \times \boldsymbol{f}).$$

Now suppose the binary star has a circular orbit with separation $\boldsymbol{r} = \boldsymbol{a}$. Show that the gravitational potential $\phi(\boldsymbol{x})$, where $\boldsymbol{x}$ is measured from the centre of mass of the system and $|\boldsymbol{x}| = x \gg |\boldsymbol{a}| = a$, may be written as

$$\phi(\boldsymbol{x}) \approx -\frac{GM}{x} - \frac{G\mu a^2}{2x^3}(3\cos^2\theta - 1),$$

where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{a}$.

The binary star has a distant companion of mass $M_3 \ll M$ in a circular orbit in the same plane as $\boldsymbol{a}$. Show that the energy and angular momentum of its orbit remain constant on a time-scale long compared with the orbital period of the inner binary.

Find the orbital period of star 3 about the close binary and comment on the consequences of mass transfer between its components.

2

What are the essential characteristics of a cataclysmic variable?

Describe the observed period distribution $N(P)\mathrm{d}P$, the number of cataclysmic variables with orbital periods between P and $P + \mathrm{d}P$.

A typical cataclysmic variable consists of a red dwarf of mass M_2 and a white dwarf of mass M_1 in a circular orbit with separation a. Suppose that, when in thermal equilibrium, the red dwarf obeys a radius–mass relation $R_2 \propto M_2$ and that, as long as the mass ratio $q = M_2/M_1 < 0.8$, the radius of the Roche lobe of star 2 obeys

$$\frac{R_\mathrm{L}}{a} \propto \left(\frac{M_2}{M}\right)^{\frac{1}{3}},$$

where $M = M_1 + M_2$. Show that the orbital period $P \propto M_2$ independent of M_1.

The adiabatic response of the red dwarf to rapid mass loss is to expand such that $R_2 \propto M_2^{-1/3}$. Assuming that the spin angular momentum of the two stars can be neglected, deduce that conservative mass transfer is dynamically unstable if $q < q_\mathrm{crit}$, a constant to be determined.

Discuss the physical processes responsible for the evolution of cataclysmic variables. In particular describe, without detailed calculation, the operation of magnetic braking and the concept of an Alfvén radius R_A, giving a rough relation between stellar mass-loss rate $\dot{M} \ll \dot{M}_2$, wind density ρ, wind speed v_wind and the magnetic field strength B at the Alfvénic surface.

Now suppose that the system is losing angular momentum at a rate

$$\dot{J} = -\alpha J,$$

on a time-scale much longer than the dynamical time-scale of star 2. Show that its orbital period evolves according to

$$\frac{\dot{P}}{P} = \frac{3\alpha}{3q - 4}.$$

Explain how the concept of interrupted magnetic braking can explain the cataclysmic variable period gap and what this means for the magnetic braking just above the gap.

3 Write an essay on Algol binary stars. Include a brief historical introduction culminating in a statement of the Algol paradox and its solution. Describe a typical system and show the evolution of the two components in a Hertzsprung–Russell diagram. Discuss the future evolution of an Algol system. Explain why all Algols are seen with mass ratios less than 0.7. What can you say about the origin of Algol-like systems that are so wide that the donor star could not have filled its Roche lobe before evolving to a red giant?

END OF PAPER