UNIVERSITY OF
CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 9:00 am to 12:00 pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Consider a dust particle that is undergoing migration towards its host star of mass $M_\star$ due to Poynting-Robertson drag causing its semimajor axis (a) and eccentricity (e) to evolve as

$$\begin{aligned}(da/dt)_{\rm pr} &= -Aa^{-1}(2+3e^2)(1-e^2)^{-3/2},\\ (de/dt)_{\rm pr} &= -(5/2)Aa^{-2}e(1-e^2)^{-1/2},\end{aligned}$$

where A is a constant. Find expressions for $(da/dt)_{\rm pr}$ and $(de^2/dt)_{\rm pr}$ keeping only terms up to second order in eccentricity.

(ii) The particle encounters the $(p+q):p$ resonance of an interior coplanar planet that is on a circular orbit around the star, where p and q are positive integers. This results in an additional perturbation to the particle's orbital elements of

$$\begin{aligned}(da/dt)_{\rm res} &= -2(p+q)e^q aC\sin\phi,\\ (de/dt)_{\rm res} &= -qe^{q-1}C\sin\phi,\end{aligned}$$

where C is a constant and $\phi = (p+q)\lambda - p\lambda_{\rm p} - q\varpi$ is the resonant argument in which $\lambda_{\rm p}$ and λ are the mean longitudes of the planet and particle respectively, and ϖ is the particle's longitude of pericentre. Show that the particle can only become trapped in the resonance if $e > e_{\rm min}$, where $e_{\rm min}$ should be determined assuming that $e_{\rm min} \ll 1$.

(iii) Assuming that the particle does become trapped at time $t = 0$ with an eccentricity e_0, using the expressions from (i) show that the eccentricity continues to evolve as

$$e = \sqrt{B + [e_0^2 - B]\exp\left(-t/\tau\right)},$$

where B and τ should be determined.

(iv) Determine the value of the resonant argument about which the particle's orbit will librate once its eccentricity has reached $e_{\rm max} = \sqrt{B}$.

(v) Describe how the resonant argument determines how far the planet is from the particle when it reaches its pericentre.

(vi) Determine the rate of azimuthal motion of the particle at pericentre when $e = e_{\rm max}$, and derive a condition on p and q for which this rate exceeds that of the planet. You may assume that radiation pressure can be neglected when determining the location of the resonance.

(vii) Sketch the orbit of the particle when $e = e_{\rm max}$ for the $5:3$ resonance in the frame rotating with the planet, explaining your reasoning and showing the angle ϕ on this plot.

2

(i) Consider a planetesimal belt that contains a total mass $M_{\rm tot}$ within a torus at a distance r from the star of radial width dr and vertical height $2I_{\rm max}r$. The planetesimals have a density ρ and diameters D in the range $D_{\rm min}$ to $D_{\rm max}$ with a single power law size distribution in which $n(D)dD$ is the number of bodies in the range D to $D+dD$ where $n(D) = KD^{-\alpha}$ and α is a constant. For reasonable assumptions which should be stated, determine K in terms of the aforementioned parameters.

(ii) The planetesimals' dispersal threshold $Q^\star_{\rm D} = Q_b D^b$, where Q_b and b are constants. The relative velocity of encounters is also size dependent so that $v_{\rm rel} = v_{\rm p}D^p$, where $v_{\rm p}$ and p are constants. Determine the minimum size of impacting planetesimals that cause catastrophic collisions with planetesimals of size D assuming that gravitational focussing can be ignored.

(iii) Repeat the calculation in (ii) for planetesimals that are large enough for gravitational focussing to dominate, making further assumptions if necessary to simplify the expression.

(iv) Ignoring gravitational focussing, derive an approximate expression for the rate of catastrophic collisions as a function of planetesimal size, again stating any further assumptions.

(v) Show that in steady state the rate of mass loss from bins that are logarithmically spaced in planetesimal size is the same for all bins, stating the assumptions required for this result to hold.

(vi) Hence, using the result in (iv), show that in steady state the size distribution is given by $\alpha = (21 + b + p)/(6 + b - 2p)$.

3

(i) Consider a binary comprised of two planetesimals of mass m_1 and m_2. In addition to their mutual gravity, the two bodies experience different forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Derive the equation of relative motion

$$\ddot{\mathbf{r}} + G(m_1 + m_2)\mathbf{r}/r^3 = \mathbf{F}_2/m_2 - \mathbf{F}_1/m_1,$$

where $\mathbf{r}$ is the vector from m_1 to m_2 and r its magnitude.

(ii) Define a coordinate system $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ with its origin at the centre of mass with $\hat{\mathbf{x}}$ pointing in the direction of m_2 at time $t = 0$. The binary is on a circular orbit about its centre of mass with a separation a and the motion of m_2 at $t = 0$ is in the $\hat{\mathbf{y}}$ direction. Each of the planetesimals experiences a drag force due to its motion relative to gas that is moving at a velocity $v_{\rm g}$ in the $\hat{\mathbf{y}}$ direction. The drag force is characterised by stopping times of $t_{\rm s1}$ and $t_{\rm s2}$ that can be assumed to be constant (i.e., independent of relative velocity). Determine the components of $\mathbf{F}_2/m_2 - \mathbf{F}_1/m_1$ in the $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ directions.

(iii) Rewrite the result of (ii) using the $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ coordinate system, where $\hat{\mathbf{r}}$ is in the direction from m_1 to m_2, and $\hat{\boldsymbol{\theta}}$ is orthogonal to this. Hence, or otherwise, derive the rate at which the binary orbit shrinks da/dt. You may assume that for a 2-body orbit $v^2 - 2\mu/r = -\mu/a$.

(iv) Averaging over the binary orbit, show that the binary separation is given by $a = a_0 \exp(-t/\tau)$, where a_0 is its initial separation and τ should be determined.

(v) The centre of mass of the binary considered above is on a circular orbit around a star of mass $M_\star \gg (m_1 + m_2)$ at a separation $a_{\rm b}$. The binary experiences drag due to its motion relative to circumstellar gas that is on a circular orbit at a velocity $(1-\eta)v_{\rm k}$, where $\eta \ll 1$ and $v_{\rm k}$ is the local Keplerian velocity. By considering the motion of the centre of mass of the binary, and applying the previous results, show that $da_{\rm b}/dt = -a_{\rm b}/\tau_{\rm b}$, where $\tau_{\rm b}$ should be determined.

(vi) Show that $\tau/\tau_{\rm b} \ll 1$, where you may assume that $m_1 \gg m_2$ and $t_{\rm s} \propto m^k$, where $0 < k < 1$. Comment on the implications for the evolution of the binary.

4

(i) Consider a planetary system comprised of 3 planets on circular orbits around a star of mass $M_\star$. The planets' masses are $M_j \ll M_\star$, and their semimajor axes a_j, for $j = 1, 2, 3$, where $a_1 < a_2 < a_3$. The orbital planes of the planets are defined by their complex eccentricities $y_j = I_j \exp(i\Omega_j)$, where $i = \sqrt{-1}$. These orbital planes evolve due to the planets' mutual secular interactions so that $\dot{\mathbf{y}} = i\mathbf{B}\mathbf{y}$, where the vector $\mathbf{y} = [y_1, y_2, y_3]$, and $\mathbf{B}$ is a 3x3 matrix. Derive the solution

$$y_j = \sum_{k=1}^{3} I_{jk} \exp(i[\lambda_k t + \gamma_k]),$$

explaining how I_{jk}, λ_k and γ_k are determined by the properties of $\mathbf{B}$ and initial conditions.

(ii) The matrix $\mathbf{B}$ has elements

$$\begin{aligned} B_{jk} &= 0.25 n_j (M_k/M_\star) \alpha_{jk} \bar{\alpha}_{jk} b^1_{3/2}(\alpha_{jk}), \\ B_{jj} &= -\sum_{k=1, k\neq j}^{3} B_{jk}, \end{aligned}$$

where n_j is the mean motion of planet j, $\alpha_{jk} = \bar{\alpha}_{jk} = a_j/a_k$ for $a_k > a_j$ but $\alpha_{jk} = a_k/a_j$ and $\bar{\alpha}_{jk} = 1$ otherwise, and $b^1_{3/2}(\alpha_{jk})$ is a Laplace coefficient. Determine the eigenvalues of $\mathbf{B}$ in terms of the 6 elements B_{jk} with $j \neq k$.

(iii) By definition $b^j_s(\alpha) = \pi^{-1} \int_0^{2\pi} \cos(jx)[1 - 2\alpha\cos(x) + \alpha^2]^{-s} dx$. Show that to first order in α, $b^1_{3/2}(\alpha) \approx 3\alpha$.

(iv) The planetary system architecture is such that the inner two planets are closely separated and the outer planet is much further out, i.e., $a_3 \gg a_2$. The masses of the planets are all comparable. Determine the elements B_{jk} in terms of B_{12}, $\alpha = a_1/a_2$, $\beta = a_1/a_3$, ratios of the planets' angular momenta L_i, and ratios of their masses. Hence for the above architecture determine the relative magnitude of each element.

(v) Show that two of the eigenvalues of $\mathbf{B}$ have magnitudes of approximately $B_{12}(1 + L_1/L_2)$ and $B_{13}[(L_1/L_2) + \alpha^{-3/2}]/(1 + L_1/L_2)$.

(vi) The two inner planets are initially coplanar, and the outer planet is on an orbit inclined by ΔI with respect to that plane. Derive that the inclination of the outer planet with respect to the invariable plane is initially

$$I_3 = \arctan[\sin\Delta I/(A + \cos\Delta I)],$$

where $A = L_3/(L_1 + L_2)$.

END OF PAPER