UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2019 1:30 pm to 4:30 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{4}$$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{5}$$

Conservation laws for momentum

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \hat{\Pi} = 0, \quad \hat{\Pi}_{ij} = \rho u_i u_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi}, \tag{6}$$

and energy

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{u^2}{2} + e \right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \left[\rho \mathbf{u} \left(\frac{u^2}{2} + h \right) + c \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \right] = 0. \tag{7}$$

You may assume that for any scalar function f

$$\nabla f = \frac{\partial f}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z \quad \textit{(cylindrical coordinates)} \tag{8}$$

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi \quad \textit{(spherical coordinates).} \tag{9}$$

You may assume that for any vector $\mathbf{C}$

$$(\nabla \times \mathbf{C}) \times \mathbf{C} = (\mathbf{C} \cdot \nabla)\mathbf{C} - \frac{1}{2} \nabla \left(|\mathbf{C}|^2 \right), \tag{10}$$

and in cylindrical coordinates

$$\nabla \cdot \mathbf{C} = \frac{1}{R} \frac{\partial (R C_R)}{\partial R} + \frac{1}{R} \frac{\partial C_\phi}{\partial \phi} + \frac{\partial C_z}{\partial z}, \tag{11}$$

$$\nabla \times \mathbf{C} = \left(\frac{1}{R} \frac{\partial C_z}{\partial \phi} - \frac{\partial C_\phi}{\partial z} \right) \mathbf{e}_R + \left(\frac{\partial C_R}{\partial z} - \frac{\partial C_z}{\partial R} \right) \mathbf{e}_\phi + \frac{1}{R} \left[\frac{\partial (R C_\phi)}{\partial R} - \frac{\partial C_R}{\partial \phi} \right] \mathbf{e}_z. \tag{12}$$

For any two vectors $\mathbf{C}$ *and* $\mathbf{D}$

$$\nabla \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}(\nabla \cdot \mathbf{D}) + (\mathbf{D} \cdot \nabla)\mathbf{C} - \mathbf{D}(\nabla \cdot \mathbf{C}) - (\mathbf{C} \cdot \nabla)\mathbf{D}, \tag{13}$$

$$\nabla \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{D} \cdot (\nabla \times \mathbf{C}) - \mathbf{C} \cdot (\nabla \times \mathbf{D}). \tag{14}$$

You may refer to these formulae in your solutions, but, please, make sure to provide sufficient details when using them.

1

In a magnetohydrodynamic (MHD) flow the density of cross-helicity h_c is defined as $h_c = \mathbf{u} \cdot \mathbf{B}$.

(a) Show that in ideal MHD the evolution of h_c can be described by an equation in the following form:

$$\frac{\partial}{\partial t}(\mathbf{u} \cdot \mathbf{B}) + \nabla \cdot \mathbf{F} = Q_h,$$

without making assumptions about the spatial distribution of entropy s. Here $\mathbf{F}$ is the flux of cross-helicity and Q_h is the source term (contributions that cannot be absorbed in $\nabla \cdot \mathbf{F}$). Derive an explicit expression for $\mathbf{F}$.

(b) Derive an explicit expression for Q_h and state the conditions under which $Q_h = 0$.

(c) Find the condition under which cross-helicity is conserved in the Lagrangian sense in a steady-state, isentropic flow, i.e.

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{B}) = 0.$$

State this condition in terms of the magnetic field $\mathbf{B}$, velocity $\mathbf{u}$, gravitational potential Φ and thermodynamic properties of the flow.

2

(a) Consider a one-dimensional flow in a polytropic gas with adiabatic index γ. A stationary shock at $z = 0$ separates region 1 ($z > 0$, where $\rho = \rho_1$, $p = p_1$ and $u = u_1$) from region 2 ($z < 0$, where $\rho = \rho_2$, $p = p_2$ and $u = u_2$). Derive the jump conditions across the shock (Rankine-Hugoniot relations) in the form

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2, \\ p_1 + \rho_1 u_1^2 &= p_2 + \rho_2 u_2^2, \\ u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma p_1}{\gamma - 1} \right) &= u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma p_2}{\gamma - 1} \right). \end{aligned}$$

Provide physical motivation for these relations.

(b) Use these jump conditions to derive the expression for the density ratio $D = \rho_2/\rho_1$ in terms of the pressure ratio $P = p_2/p_1$.

(c) Using the results of part (b) compute $[s]/c_v$ in terms of P. Here $[s] = s_2 - s_1$ is the entropy difference before (s_1) and after (s_2) the shock, and c_v is the specific heat capacity.

(d) Consider the limit of a *weak shock* such that $P = 1 + \delta$, $\delta \ll 1$. Calculate $[s]/c_v$ in this limit to the lowest non-zero order in δ.

(e) Use the results of part (d) to determine which would raise the gas entropy more: (i) a single strong shock with pressure ratio $P_s \gg 1$ or (ii) multiple successive weak shocks resulting in the same ultimate increase of pressure (i.e. the pressure after passage of the last weak shock divided by the pressure before the passage of the first weak shock is again equal to $P_s \gg 1$). For simplicity assume that all weak shocks have the same pressure ratio $P_w = 1 + \delta$, $\delta \ll 1$.

3 Consider an axisymmetric magnetostatic configuration of ideal plasma, in which both thermal pressure p as well as stresses due to magnetic field $\mathbf{B}$ play important roles. This configuration has a cylindrical symmetry such that all its characteristics are independent of the z-coordinate along the axis of rotational symmetry. Gravity can be ignored for this problem.

(a) Describe the behavior of the radial (cylindrical R) components, B_R and j_R, of the magnetic field and current density in this configuration, assuming regularity of all variables on the z axis.

(b) Derive a (differential) equation relating the thermal pressure p, the z-component of the magnetic field B_z, and the current $I(R)$ enclosed within radius R,

$$I(R) = 2\pi \int_0^R j_z R dR, \tag{1}$$

(where j_z is the z-component of the current density) for this configuration.

(c) Assume now that B_z is constant everywhere in space, and that

$$p(R) = p_0 \left(1 + \frac{R^2}{a^2}\right)^{-k}, \tag{2}$$

where p_0, a, k are constants. Using the equation derived in part (b), compute the profile of $I(R)$. What condition should be imposed on k for the current $I(R)$ to remain finite as $R \to \infty$?

(d) Now assume instead that vertical current is zero everywhere, $I(R) = 0$. Assume pressure behaviour in the form

$$p(R) = p_0 \exp\left(-R^2/a^2\right), \tag{3}$$

and that $B_z(R = 0) = 0$. Using again the equation derived in part (b), determine the radial profiles of B_z and of the toroidal component of the current density j_ϕ. What is the value of $j_\phi(R \to 0)$?

4 Consider accretion of ionized gas from the interstellar medium with density ρ_0 and sound speed c_0 onto a magnetized, non-rotating, and non-moving neutron star of mass $M_\star$ and radius $R_\star$. The magnetic field has a dipolar geometry, i.e.

$$\mathbf{B} = \frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r} - \mathbf{m}r^2}{r^5}, \tag{1}$$

where $\mathbf{m}$ is the vector magnetic moment of the neutron star. The magnetic field of the neutron star is very strong and can be considered as unperturbed by external stresses in the region of interest. It regulates the accretion of gas to occur only along the field lines very close to the magnetic axis, which subtend the angle $\theta < \theta_\star \ll 1$ in spherical coordinates aligned with the magnetic axis (only these field lines get loaded with the gas at large distances, far outside the region of interest).

Assume that the gas is isentropic, with pressure p and density ρ related via $p = K\rho^\gamma$, when K and γ are constants. Upon reaching the neutron star surface the accreted gas gets freely absorbed (i.e. there is no back reaction on the incoming flow).

(a) By solving for the shape of field lines, or otherwise, determine how the cross-section of the flux tube along which accretion occurs changes as a function of distance from the neutron star centre. You may assume that you stay close to the magnetic axis.

(b) Argue that, when $\theta_\star \ll 1$, gas accretion along field lines can be considered as a one-dimensional hydrodynamical problem. Formulate the equations describing this problem in steady-state, accounting for the gas pressure and neutron star gravity. For simplicity, assume that magnetic stresses can be neglected.

(c) By analysing the equations derived in part (b), demonstrate that they adopt transonic solutions for γ less than some critical value, which you should determine.

(d) In the transonic case, determine the distance to the sonic surface as well as the values of the sound speed and density at this location. Calculate the mass accretion rate $\dot{M}$ onto the neutron star in this setup, i.e. along the field lines loaded with gas.

(e) Using the results obtained so far, suggest a constraint on $\theta_\star$ that has to be satisfied for the transonic flow to be described by the one-dimensional equations derived in part (b).

END OF PAPER