UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, MAAS

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2019 1:30 pm to 4:30 pm

PAPER 312

ADVANCED COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) According to the in-in formalism, the leading-order correction to the expectation value of an operator Q is given by

$$\langle Q(\tau)\rangle = \Re\left\langle -2\mathrm{i}Q^{\mathrm{I}}(\tau)\int_{-\infty(1-\mathrm{i}\varepsilon)}^{\tau} a(\tau')H^{\mathrm{I}}_{\mathrm{int}}(\tau')\,\mathrm{d}\tau'\right\rangle\,, \qquad (\dagger)$$

where a is the scale factor and τ is the conformal time (defined by $\mathrm{d}t = a\,\mathrm{d}\tau$). Describe the vacuum state for the in-in formalism and discuss what distinguishes the in-in formalism from the S-matrix in-out calculation. Splitting the Hamiltonian into a background part, the free quadratic part and the cubic interacting part, which term describes the time evolution of the interaction picture fields Q^{I} and $H^{\mathrm{I}}_{\mathrm{int}}$?

(ii) Consider a scalar field ϕ, minimally coupled to gravity, with a non-standard kinetic term parametrized by the function $P(X,\phi)$ in the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + P(X,\phi)\right]\,,$$

where R is the Ricci scalar and $2X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the kinetic term. The background equations of motion can be derived from a variation of the above action with respect to the metric and are given by the Friedmann equations relating the Hubble rate H to pressure $\bar{p}$ and density $\bar{\rho}$

$$3H^2 = \bar{\rho} = 2\bar{X}P_{,\bar{X}} - P\,, \qquad \dot{H} = -\frac{1}{2}(\bar{\rho}+\bar{p}) = -\bar{X}P_{,\bar{X}}\,.$$

Here $P_{,\bar{X}}$ is the derivative of $P(X,\phi)$ with respect to X in the background. The corresponding slow-roll parameter is thus given by $\epsilon = -\dot{H}/H^2 = \bar{X}P_{,\bar{X}}/H^2$.

(ii,a) Show that the speed of sound is given by

$$c_s^2 = \frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{\rho}} = \frac{P_{,\bar{X}}}{P_{,\bar{X}} + 2XP_{,\bar{X}\bar{X}}}\,.$$

Hint: Rewrite the derivative in terms of $\mathrm{d}\bar{p}/\mathrm{d}\bar{X}$ and $\mathrm{d}\bar{\rho}/\mathrm{d}\bar{X}$.

(ii,b) Perturb the scalar field ϕ around a homogeneous and isotropic background as $\phi(\boldsymbol{x},t) = \bar{\phi}(t) + \delta\phi(\boldsymbol{x},t)$. For interesting levels of observable non-Gaussianity, scalar field perturbations have to dominate over metric perturbations. In the following we will thus continue to assume an unperturbed background $\delta R = 0$ and $g_{\mu\nu} = \mathrm{diag}(-1,a^2,a^2,a^2)$. Given these simplifications, show that we have for the kinetic term

$$X = \bar{X} + \delta X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}\dot{\bar{\phi}}^2 + \dot{\bar{\phi}}\,\delta\dot{\phi} + \frac{1}{2}\delta\dot{\phi}^2 - \frac{1}{2a^2}(\partial_i\delta\phi)^2\,.$$

Express the inflaton fluctuations in the above expression in terms of the curvature perturbation $\zeta = -(H/\dot{\bar{\phi}})\delta\phi$ ignoring the slow-roll suppressed derivatives of H and $\dot{\bar{\phi}}$, to obtain

$$\delta X = \frac{\bar{X}}{H^2}\left[-2H\dot{\zeta} + \dot{\zeta}^2 - \frac{1}{a^2}(\partial_i\zeta)^2\right]\,.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

Expand the function $P(X,\phi)$ in the action to third order in δX as

$$P(X) = P(\bar{X}) + P_{,\bar{X}}\delta X + \frac{1}{2!}P_{,\bar{X}\bar{X}}\delta X^2 + \frac{1}{3!}P_{,\bar{X}\bar{X}\bar{X}}\delta X^3$$

and show that the cubic terms in the action can be written as

$$S_3 = \int \mathrm{d}^4x \, \frac{a^3\epsilon}{H}\frac{1-c_s^2}{c_\mathrm{s}^2}\left[\frac{1}{a^2}\dot{\zeta}(\partial_i\zeta)^2 + \mathcal{A}\dot{\zeta}^3\right] ,$$

where

$$\mathcal{A} = -1 - \frac{2}{3}\bar{X}\frac{P_{,\bar{X}\bar{X}\bar{X}}}{P_{,\bar{X}\bar{X}}} .$$

(ii,c) Performing the Legendre transformation from the Lagrange function, the Hamiltonian $H^\mathrm{I}_\mathrm{int}$ is given by $-L^\mathrm{I}_\mathrm{int}$. In the following, we will consider the component of the action that is proportional to $\dot{\zeta}(\partial_i\zeta)^2$:

$$H^\mathrm{I}_\mathrm{int} = -\int \mathrm{d}^3x \, \frac{a\epsilon}{H}\frac{1-c_s^2}{c_\mathrm{s}^2}\dot{\zeta}(\partial_i\zeta)^2 , \qquad (\star)$$

with slow-roll parameter ϵ (which you may assume is effectively constant) and scale factor given by $a = -1/(H\tau)$ with Hubble constant H. Here, in the interaction picture, the linear density perturbation ζ is a Gaussian random field with two-point correlator,

$$\left\langle \zeta(\boldsymbol{k})\zeta(\boldsymbol{k}')\right\rangle = (2\pi)^3 u_k u_k^* \delta^{(\mathrm{D})}(\boldsymbol{k}+\boldsymbol{k}') , \qquad (\#)$$

where the mode functions $u_k(\tau)$ and their conformal time derivatives are given by

$$u_k(\tau) = \frac{H}{\sqrt{4\epsilon c_\mathrm{s} k^3}}\left(1 + \mathrm{i}kc_\mathrm{s}\tau\right)e^{-\mathrm{i}kc_\mathrm{s}\tau} , \quad \text{and} \quad u_k'(\tau) = \frac{H}{\sqrt{4\epsilon c_\mathrm{s} k^3}}c_\mathrm{s}^2 k^2 \tau e^{-\mathrm{i}kc_\mathrm{s}\tau} .$$

Use Wick's theorem, together with the power spectrum (#) and the in-in formalism expression (†), to show that the three-point correlator of ζ for the interaction Hamiltonian ($\star$) can be written in the following form

$$\left\langle \zeta(\boldsymbol{k}_1,0)\zeta(\boldsymbol{k}_2,0)\zeta(\boldsymbol{k}_3,0)\right\rangle = (2\pi)^3\delta^{(\mathrm{D})}\left(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3\right)\frac{1-c_s^2}{c_s^3}\frac{H^4}{32\epsilon^2}\frac{1}{(k_1k_2k_3)^3}$$
$$\times\Re\left\{-\mathrm{i}\int_{-\infty(1-\mathrm{i}\varepsilon)}^{0}\mathrm{d}\tau \, e^{-iKc_\mathrm{s}\tau}\left[k_1^2(k_1^2-k_2^2-k_3^2)\left(1+\mathrm{i}k_2c_\mathrm{s}\tau\right)\left(1+\mathrm{i}k_3c_\mathrm{s}\tau\right) + 2\text{ cyc.}\right]\right\} ,$$

with $K = k_1 + k_2 + k_3$. You do not need to evaluate the time integral explicitly.

2 Consider the perturbative expansion of the matter density field $\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \cdots$, where in Fourier space the n-th order contribution is given by

$$\delta^{(n)}(\boldsymbol{k}, t) = D^n(t) \prod_{i=1}^{n} \left\{ \int \frac{\mathrm{d}^3 q_n}{(2\pi)^3} \delta_{\mathrm{lin}}(\boldsymbol{q}_i) \right\} F_n(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)(2\pi)^3 \delta^{(\mathrm{D})}(\boldsymbol{k} - \boldsymbol{q}_1 - \cdots - \boldsymbol{q}_n) \,.$$

Here, δ_{lin} is the underlying Gaussian density field, with two-point correlator $\langle \delta_{\mathrm{lin}}(\boldsymbol{k}) \delta_{\mathrm{lin}}(\boldsymbol{k}') \rangle = (2\pi)^3 \delta^{(\mathrm{D})}(\boldsymbol{k} + \boldsymbol{k}') P_{\mathrm{lin}}(k)$, F_n are the gravitational coupling kernels (satisfying $F_1(\boldsymbol{q}_1) \equiv 1$) and $D(t)$ is the linear growth factor normalized to unity at the present time.

(i) Draw the Feynman diagrams for the two contributions to the one-loop power spectrum and write down the corresponding expressions in terms of the gravitational coupling kernels F_n. Argue why the odd 1-2 contribution has to vanish. Discuss the behaviour of the terms when the loop momentum q is much larger than the external momentum k and write down the appropriate leading-order counter term in the effective field theory of large-scale structure.

(ii) Calculate the leading-order skewness of the density field $\left\langle \delta^3(\boldsymbol{x}) \right\rangle$ given that the symmetrized form of the second-order coupling kernel is

$$F_{2,s}(\boldsymbol{q}_1, \boldsymbol{q}_2) = \frac{5}{7} + \frac{1}{2}\mu \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7}\mu^2 \,,$$

where $\mu = \hat{\boldsymbol{q}}_1 \cdot \hat{\boldsymbol{q}}_2$ is the cosine of the enclosed angle. Express the result in terms of the variance of the field $\sigma^2 = \left\langle \delta^2(\boldsymbol{x}) \right\rangle$.

(iii) Parameterize the skewness as $\left\langle \delta^3 \right\rangle = S_3 \sigma^4$. Assume that for weakly non-Gaussian fields, the one-point moment generating function for the density field $\delta(\boldsymbol{x})$ at a single position $\boldsymbol{x}$ can be written as

$$\mathcal{M}(J) = \langle \exp\left[\delta J\right] \rangle = \exp\left[\frac{1}{2} J^2 \sigma^2\right] \left(1 + \frac{\sigma^4 S_3}{3!} J^3\right) \,. \qquad (\star)$$

Given that the probability density function (PDF) is the inverse Laplace transformation of the moment generating function

$$\mathbb{P}(\delta) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}J}{2\pi \mathrm{i}} \exp\left[-J\delta\right] \mathcal{M}(J) \,, \qquad (\dagger)$$

show that the probability density function of the weakly non-Gaussian field is given by

$$\mathbb{P}(\delta) = \left(1 - \frac{S_3 \sigma^4}{3!} \frac{\mathrm{d}^3}{\mathrm{d}\delta^3}\right) \mathbb{P}_{\mathrm{G}}(\delta) \,,$$

where $\mathbb{P}_{\mathrm{G}}(\delta)$ is the Gaussian PDF

$$\mathbb{P}_{\mathrm{G}}(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\frac{\delta^2}{\sigma^2}\right] \,.$$

Show that this result is equivalent to the Edgeworth expansion of the probability density function

$$\mathbb{P}(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\frac{\delta^2}{\sigma^2}\right] \left(1 + \frac{S_3(\delta^3 - 3\delta\sigma^2)}{3!\sigma^2}\right) \,.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

Hint: Replace the integration variable $J = \mathrm{i}y$ in Eq. (†), write J^3 in Eq. ($\star$) as a derivative of the exponential, and use the Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \; e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \, .$$

3

(i) The dimensionless temperature anisotropy of the CMB is $\Theta(\eta, \mathbf{x}, \mathbf{e})$, where η is conformal time, $\mathbf{x}$ is comoving position and $\mathbf{e}$ is the photon propagation direction. For scalar perturbations in the conformal Newtonian gauge, the Boltzmann equation for Θ is

$$\dot{\Theta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta + \mathbf{e}\cdot\boldsymbol{\nabla}\psi - \dot{\phi} = \dot{\tau}\left(\Theta - \Theta_{00}/\sqrt{4\pi}\right) - \frac{\dot{\tau}}{10}\sum_{|m|\leqslant 2}\left(\Theta_{2m} - \sqrt{6}E_{2m}\right)Y_{2m}(\mathbf{e}) - \dot{\tau}\mathbf{e}\cdot\mathbf{v}_b\,,$$

where overdots denote partial differentiation with respect to conformal time, ϕ and ψ are the metric potentials, and τ is the optical depth to Thomson scattering. The scattering terms involve the spherical multipoles Θ_{lm} of the temperature anisotropy, a polarization correction from the $l = 2$ multipoles of the E-mode polarization, E_{2m}, and the baryon peculiar velocity $\mathbf{v}_b$. Explain why the spatial Fourier transform of the temperature anisotropy, $\Theta(\eta, \mathbf{k}, \mathbf{e})$, and E-mode polarization, $E(\eta, \mathbf{k}, \mathbf{e})$, is axisymmetric about the wavevector $\mathbf{k}$.

Expanding in Legendre polynomials as

$$\Theta(\eta, \mathbf{k}, \mathbf{e}) = \sum_{l\geqslant 0}(-i)^l \Theta_l(\eta, \mathbf{k}) P_l(\hat{\mathbf{k}}\cdot\mathbf{e})\,,$$

and similarly for $E(\eta, \mathbf{k}, \mathbf{e})$, derive the Boltzmann hierarchy

$$\dot{\Theta}_l + k\left(\frac{l+1}{2l+3}\Theta_{l+1} - \frac{l}{2l-1}\Theta_{l-1}\right) = -\dot{\tau}\left[(\delta_{l0} - 1)\,\Theta_l - \delta_{l1}v_b + \frac{1}{10}\delta_{l2}\left(\Theta_2 - \sqrt{6}E_2\right)\right] + \delta_{l0}\dot{\phi} + \delta_{l1}k\psi\,. \quad (*)$$

Here, the Fourier transform of the baryon velocity is $\mathbf{v}_b = i\hat{\mathbf{k}}v_b$ and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$.

[*You may wish to use* $(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu)$.]

(ii) Consider scales small enough that the effects of cosmic expansion and gravity can be ignored. Write down the $l = 0$ and $l = 1$ moments of the Boltzmann hierarchy $(*)$ in this limit and give physical interpretations of these equations.

Given that the bulk velocity of the CMB is $\mathbf{v}_\gamma(\eta, \mathbf{k}) = i\hat{\mathbf{k}}v_\gamma(\eta, \mathbf{k})$, with $v_\gamma = -\Theta_1$, by considering the exchange of momentum between the baryons and CMB show that

$$\dot{v}_b \approx -\frac{\dot{\tau}}{R}(v_\gamma - v_b)\,,$$

where you should specify R in terms of the (unperturbed) energy densities of the CMB and baryons.

[*You may assume that the Euler equation for a non-interacting, pressure-free fluid is* $\dot{v} + \frac{\dot{a}}{a}v + k\psi = 0$.]

[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Explain briefly what is meant by the tight-coupling approximation.

By considering the slip velocity $v_\gamma - v_b$, or otherwise, show that, to first order in $k|\dot{\tau}|^{-1}$,

$$(1+R)\dot{\Theta}_1 + \frac{2}{5}k\Theta_2 - k\Theta_0 \approx -\frac{R^2}{1+R}k\dot{\tau}^{-1}\dot{\Theta}_0 \,.$$

Using the $l = 2$ moment of $(*)$, and the equivalent for the E-mode polarization,

$$\dot{E}_l + k\left(\frac{\sqrt{(l+1)^2-4}}{2l+3}E_{l+1} - \frac{\sqrt{l^2-4}}{2l-1}E_{l-1}\right) = \dot{\tau}\left[E_l - \delta_{l2}\frac{3}{5}\left(E_2 - \frac{1}{\sqrt{6}}\Theta_2\right)\right] \,,$$

show further that $\Theta_2 \approx -\frac{8}{9}k\dot{\tau}^{-1}\Theta_1$.

Hence show that Θ_0 satisfies the oscillator equation

$$\ddot{\Theta}_0 + \frac{k^2|\dot{\tau}^{-1}|}{3(1+R)}\left(\frac{R^2}{1+R} + \frac{16}{15}\right)\dot{\Theta}_0 + \frac{k^2}{3(1+R)}\Theta_0 \approx 0 \,.$$

Comment, briefly, on the observational implications of the damping term in this equation.

4 Consider the dynamics of freely-propagating massless neutrinos in a spatially-flat universe with small tensor perturbations (gravitational waves). The line element is

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right] ,$$

where $a(\eta)$ is the scale factor at conformal time η. The metric perturbation h_{ij} is symmetric and trace-free, and has vanishing divergence. Throughout this question you should work to first order in the perturbation h_{ij}.

(i) Let $\epsilon = aE$ be the comoving energy of a neutrino measured by an observer at constant x^i, and $\mathbf{e}$ be the propagation direction defined relative to an orthonormal frame of vectors (which would be aligned with the coordinate directions if h_{ij} were vanishing). Write down the neutrino stress–energy tensor in terms of the one-particle distribution function $f(\eta, \mathbf{x}, \epsilon, \mathbf{e})$ and the neutrino 4-momentum.

Writing f as the sum of a background part and a perturbation,

$$f(\eta, \mathbf{x}, \epsilon, \mathbf{e}) = \bar{f}(\epsilon) - \Theta(\eta, \mathbf{x}, \mathbf{e})\frac{d\bar{f}}{d\ln\epsilon} ,$$

show that the orthonormal-frame components of the anisotropic stress are

$$\Pi^{\hat{\imath}\hat{\jmath}}(\eta, \mathbf{x}) = -4\bar{\rho}_\nu \int \frac{d\mathbf{e}}{4\pi}\, \Theta(\eta, \mathbf{x}, \mathbf{e}) \left(e^{\hat{\imath}} e^{\hat{\jmath}} - \frac{1}{3}\delta^{\hat{\imath}\hat{\jmath}}\right) ,$$

where $\bar{\rho}_\nu$ is the (unperturbed) neutrino energy density.

(ii) Given that the comoving energy of a photon evolves as

$$\frac{d\ln\epsilon}{d\eta} = -\frac{1}{2}\dot{h}_{ij} e^{\hat{\imath}} e^{\hat{\jmath}} ,$$

where overdots denote partial differentiation with respect to η, derive the collisionless Boltzmann equation for $\Theta(\eta, \mathbf{x}, \mathbf{e})$.

Assuming that $\Theta = 0$ at $\eta = 0$, show that

$$\Theta(\eta, \mathbf{x}, \mathbf{e}) = -\frac{1}{2}\int_0^\eta \dot{h}_{ij}(\eta', \mathbf{x} - \chi'\mathbf{e}) e^{\hat{\imath}} e^{\hat{\jmath}}\, d\eta' , \qquad (*)$$

where $\chi' \equiv \eta - \eta'$.

(iii) Anisotropic stress is sourced by the quadrupole ($l = 2$) part of Θ. For the case of a helicity ± 2 plane wave with wavevector $\mathbf{k} = k\hat{\mathbf{z}}$ along the z-axis, for which the metric perturbation is

$$h_{ij}(\eta, \mathbf{x}) = \frac{1}{\sqrt{2}} m_{ij}^{(\pm 2)} h^{(\pm 2)}(\eta, k\hat{\mathbf{z}}) e^{i\mathbf{k}\cdot\mathbf{x}} ,$$

with $m_{ij}^{(\pm 2)} = \frac{1}{2}(\delta_{i1} \pm i\delta_{i2})(\delta_{j1} \pm i\delta_{j2})$, show from $(*)$ that the quadrupole contribution to $\Theta(\eta, \mathbf{x}, \mathbf{e})$ is

$$\Theta(\eta, \mathbf{x}) \supset -\frac{15}{2\sqrt{2}} \left(m_{ij}^{(\pm 2)} e^{\hat{\imath}} e^{\hat{\jmath}}\right) e^{i\mathbf{k}\cdot\mathbf{x}} \int_0^\eta \dot{h}^{(\pm 2)}(\eta', k\hat{\mathbf{z}}) \frac{j_2(k\chi')}{(k\chi')^2}\, d\eta' .$$

[QUESTION CONTINUES ON THE NEXT PAGE]

[*You may wish to use* $Y_{2\,\pm 2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi}$ *and the integral*

$$\frac{1}{16}\int_{-1}^{1}\left(1-\mu^2\right)^2 e^{-ix\mu}\,d\mu = \frac{j_2(x)}{x^2}\,.\Big]$$

Hence show that the neutrino anisotropic stress is

$$\Pi^{\hat{\imath}\hat{\jmath}}(\eta,\mathbf{x}) = 2\sqrt{2}\bar{\rho}_\nu e^{i\mathbf{k}\cdot\mathbf{x}} m^{(\pm 2)\,ij}\int_0^\eta \dot{h}^{(\pm 2)}(\eta', k\hat{\mathbf{z}})\frac{j_2(k\chi')}{(k\chi')^2}\,d\eta'\,,$$

where, numerically, $m^{(\pm 2)\,ij} = m^{(\pm 2)}_{ij}$.

(iv) The metric perturbation h_{ij} evolves as

$$\ddot{h}_{ij} + 2\frac{\dot{a}}{a}\dot{h}_{ij} - \boldsymbol{\nabla}^2 h_{ij} = -16\pi G a^2 \Pi_{ij}\,,$$

where, at linear order, the coordinate components of the anisotropic stress Π_{ij} are equal to the orthonormal-frame components $\Pi^{\hat{\imath}\hat{\jmath}}$. Assuming that neutrinos are the only source of anisotropic stress, show that

$$\ddot{h}^{(\pm 2)}(\eta, k\hat{\mathbf{z}}) + 2\frac{\dot{a}}{a}\dot{h}^{(\pm 2)}(\eta, k\hat{\mathbf{z}}) + k^2 h^{(\pm 2)}(\eta, k\hat{\mathbf{z}}) =$$
$$-24\left(\frac{\dot{a}}{a}\right)^2\frac{\bar{\rho}_\nu}{\bar{\rho}_{\rm tot}}\int_0^\eta \dot{h}^{(\pm 2)}(\eta', k\hat{\mathbf{z}})\frac{j_2(k\chi')}{(k\chi')^2}\,d\eta'\,,$$

where $\bar{\rho}_{\rm tot}$ is the unperturbed total energy density.

What do you expect the qualitative effect of the neutrino anisotropic stress to be on the evolution of tensor perturbations?

END OF PAPER