UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2019 1:30 pm to 4:30 pm

PAPER 311

BLACK HOLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider an isolated uncharged star that undergoes gravitational collapse to form a black hole. Briefly explain why it is believed that the spacetime at late time is characterized by only two parameters.

(b) The metric and gauge potential of a spherically symmetric isolated charged gravitating object in five spacetime dimensions is, in Schwarzschild-like coordinates,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2)\,, \quad A = -\frac{Q}{r^2}dt\,,$$

where $\theta \in [0, \pi/2]$, $\phi \in [0, 2\pi]$, $\psi \in [0, 2\pi]$ are coordinates on a unit radius three sphere and

$$f(r) = 1 - \frac{2M}{r^2} + \frac{Q^2}{r^4}\,,$$

where $M > 0$ is the mass of the object and $Q > 0$ is the electric charge.

(i) Construct the analogue of ingoing Eddington-Finkelstein coordinates and determine the form of the metric in these coordinates. Explain briefly why this metric can be analytically extended across the surface $r = r_+$ where $r_+ = \sqrt{M + \sqrt{M^2 - Q^2}}$, provided $M \geqslant Q$.

(ii) Show that $r = r_+$ is a Killing horizon of the Killing vector field $k = \partial/\partial t$ and determine its associated surface gravity.

(iii) Sketch the Penrose diagram(s) for the spacetime(s) described by this metric for any $M > 0$ and $Q > 0$.

(iv) The equation of motion for an electrically charged scalar field can be written as

$$\mathcal{D}_a \mathcal{D}^a \Phi = 0\,,$$

where $\mathcal{D} = \nabla - i\, q\, A$ and q is the scalar electric charge. Consider Φ of the form

$$\Phi = e^{-i\,\omega\, t} R(r)\Theta(\theta)\,,$$

and show that the wave equation reduces to ordinary differential equations for $R(r)$ and $\Theta(\theta)$, which you should determine. You may assume that

$$\nabla_\mu Q^\mu = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} Q^\mu\right)\,.$$

2 Let Σ denote a three-dimensional spacelike hypersurface with unit normal n, embedded in a four-dimensional spacetime $(\mathcal{M}, g)$.

(i) Show how to construct the first fundamental form h_{ab}. Describe how to construct the second fundamental form K_{ab} of this surface, show that K_{ab} is a symmetric $2-$tensor and that it is independent of how n is extended in a neighbourhood of Σ.

(ii) Derive the Gauss equation for the components of the curvature tensor ${}^{(3)}R^a{}_{bcd}$ of Σ in terms of the four dimensional Riemann tensor and second fundamental form.

(iii) Show that ${}^{(3)}R = R + 2R_{ab}n^a n^b - K^2 + K_{ab}K^{ab}$, where R_{ab} is the Ricci tensor associated with g, R its Ricci scalar and $K = g^{ab}K_{ab}$.

(iv) Suppose that $(\mathcal{M}, g)$ satisfies the Einstein equation and that K_{ab} is proportional to the induced metric on Σ. Show that ${}^{(3)}R \geqslant 0$ provided that $24\pi\,\rho - K^2 \geqslant 0$, where $\rho \equiv T_{ab}n^a n^b$ and $K \equiv g^{ab}K_{ab}$.

3 A spacetime $(\mathcal{M}, g)$ is *stably causal* if there exists a continuous timelike vector field t^a such that the spacetime $(\mathcal{M}, \tilde{g})$ possesses no closed timelike curves, where $\tilde{g}_{ab} = g_{ab} - t_a\, t_b$. In what follows, you may assume that $\mathcal{M}$ is connected.

(i) Show that every timelike and null vector of g_{ab} is a timelike vector of $\tilde{g}_{ab}$, and thus that the light cone of $\tilde{g}_{ab}$ is strictly larger than that of g_{ab}.

(ii) Show that if g_{ab} is stably causal, then for some timelike vector field r^a, $g_{ab} - r_a r_b$ is also stably causal.

(iii) Show that in a stably causal and null geodesically complete spacetime, two points with the same chronological future are the same.

(iv) Let $(\mathcal{M}, g)$ be any spacetime where $\mathcal{M}$ is connected. Show that if a nonempty subset $\mathcal{U} \subset \mathcal{M}$ is equal to its own chronological future and also equal to its own chronological past, then $\mathcal{U} = \mathcal{M}$.

4 The marking scheme is equally distributed between (a) and (b).

(a) Write an essay giving a detailed account of the quantum theory of a free scalar field in a globally hyperbolic spacetime. You should explain carefully why the particle concept is ambiguous in general and why it can be made unambiguous in a stationary spacetime. Describe how to calculate the expected number of particles produced in a spacetime that is stationary at early and late times but time-dependent in between.

(b) Explain why the discovery that black holes emit thermal radiation implies that the laws of black hole mechanics can be reinterpreted in thermodynamical terms.

END OF PAPER