UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 9:00 am to 12:00 pm

PAPER 310

COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For an arbitrary spacetime, write down the form of the energy-momentum tensor $T_{\mu\nu}$ for a perfect fluid in any reference frame. Besides the spacetime metric, how many functions of spacetime does one need to specify in order to completely determine $T_{\mu\nu}(x)$? Write down the equations that determine the evolution of $T_{\mu\nu}$.

(b) Write down the metric of an open, closed and flat FLRW spacetime, in any coordinates you like. What is the volume of the spatial hypersurface defined by constant cosmological time t in each of the three cases? Write down explicitly the evolution equations for $T_{\mu\nu}$ in a spatially flat FLRW spacetime.

(c) Assume that the universe is flat. Write down an integral expression for the age of the universe as function of today's fractional energy densities $\Omega_i(t_0)$, with i being non-relativistic matter (a.k.a. "dust"), radiation and a cosmological constant where t_0 is today's cosmological time. Then, assuming that there is only matter, compute the age of the universe in giga years, using today's value of the Hubble parameter H_0.

(d) Recall the definition of the Hubble slow-roll parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \, . \tag{1}$$

What values of ϵ correspond to accelerated expansion and accelerated contraction, respectively? Consider a real canonical scalar field ϕ with a potential of the form $V = \lambda\phi^p$ for some positive p. For what values of ϕ are the potential slow-roll parameters ϵ_V and η_V smaller than one? Let ϕ_* be such that

$$\epsilon_V(\phi_*), \eta_V(\phi_*) \ll 1 \, . \tag{2}$$

Compute $\epsilon(\phi_*, \dot{\phi}_*)$ assuming $\dot{\phi}_* = 2\sqrt{\lambda}\phi_*^{p/2}$. Is the expansion of the universe accelerated or decelerated for these values?

2

(a) Using dimensional analysis, determine the exponents α_a in

$$\rho_{rel} = g\frac{\pi^2}{30}T^{\alpha_1}\lambda\,, \qquad \rho_{nr} = g\left(\frac{mT}{2\pi}\right)^{\alpha_2} e^{(\mu-m)/T}\left(m+\frac{3}{2}T\right)\,, \qquad (1)$$

$$n_{rel} = g\frac{\zeta(3)}{\pi^2}T^{\alpha_3}\tilde{\lambda}\,, \qquad s_{rel} = g\frac{2\pi^2}{45}T^{\alpha_4}\lambda\,, \qquad (2)$$

$$n_{nr} = g\left(\frac{mT}{2\pi}\right)^{\alpha_5} e^{(\mu-m)/T}\,, \qquad p_{nr} = g\left(\frac{mT}{2\pi}\right)^{\alpha_6} e^{(\mu-m)/T}\,T\,, \qquad (3)$$

where $\lambda = \{1, 7/8\}$, $\tilde{\lambda} = \{1, 3/4\}$ for bosons and fermions respectively, $\zeta(3) \simeq 1.2$, T is the temperature, g is the number of degenerate states, μ is the chemical potential, m is the mass, n is the number density, ρ the energy density, p the pressure, s the entropy density and the labels "*rel*" and "*nr*" indicate relativistic and non-relativistic particles respectively. In the following the labels $\{e, p, n, b, \gamma\}$ refer to electrons, protons, neutrons, baryons and photons, respectively.

(b) Write down the main reaction that keeps electrons and positrons in chemical equilibrium around $T \sim 1$ MeV. Compute n_e/n_γ as function of T for $T < m_e$ assuming chemical equilibrium and that all chemical potentials are negligible.

(c) Using the Helium abundance today compute n_p/n_b [*Hint: You may approximate* $m_{He} \simeq 4m_H$*, but do not neglect any term of order* n_n/n_p]. Then, using the baryon-to-photon number ratio today n_b/n_γ and the observed charge neutrality, estimate n_e/n_γ.

(d) Using the above results, determine before what temperature the chemical equilibrium between electron and positron must break down [*Hint: To find a numerical solution for the resulting transcendental equation, take the logarithm of the whole expression and neglect all numerical factors inside the log except for* 10^{-10}. *Recall* $\log 10 \simeq 2.3$]

3

At late times in the universe well after matter-radiation equality (and assuming $K = 0$ throughout), the gravitational potential Φ is given by the Poisson equation

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}_m \Delta_m \tag{1}$$

in terms of the comoving gauge density contrast Δ_m, the matter background density $\bar{\rho}_m$, and the scale factor a. The Einstein equation for potential evolution is

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 0, \tag{2}$$

where primes indicate derivatives with respect to conformal time τ and $\mathcal{H}$ is the conformal Hubble rate. If required, you may use without proof that, from the Friedmann equations, $\mathcal{H}' - \mathcal{H}^2 = -4\pi G a^2 \bar{\rho}_m$.

(a) From the above equations, explain why $\Phi \propto \Delta_m/a$ and hence show that the evolution equation for Δ_m, rewritten in terms of cosmological time t, is given by:

$$\ddot{\Delta}_m + 2H\dot{\Delta}_m - 4\pi G\bar{\rho}_m\Delta_m = 0, \tag{3}$$

where dots indicate derivatives with respect to t.

(b) Show that this evolution equation implies that during matter domination the density contrast grows as $\Delta_m \propto a$.

(c) Explain briefly why increasing the dark energy density $\bar{\rho}_\Lambda$ (assuming an equation of state $w = -1$ and no changes to other energy densities) reduces the growth of structure. From the evolution equation above, deduce the evolution of Δ_m during dark energy domination. You may assume that the dark energy density $\bar{\rho}_\Lambda$ obeys $\bar{\rho}_\Lambda \gg \bar{\rho}_m$.

(d) Now consider a non-standard cosmological evolution in which, at a time t_f during matter domination, new physics instantaneously modifies the relevant evolution equation (during matter domination, on subhorizon scales) to be:

$$\ddot{\Delta}_m + 2H\dot{\Delta}_m - 4\pi G\bar{\rho}_m(1-f)\Delta_m = 0 \tag{4}$$

where f is a constant, with $0 < 1 - f < 1$. The background energy densities and the background evolution are unaffected by this new physics. Show that, in this non-standard cosmology, the amplitude of Δ_m today is reduced, with respect to the amplitude in the standard cosmology, by a factor

$$\approx \left(\frac{t_\Lambda}{t_f}\right)^{s(f)-\frac{2}{3}}, \tag{5}$$

where you should determine the function $s(f)$, and t_Λ is the time at which dark energy domination begins.

(e) A ΛCDM cosmology with a higher dark energy density could give the same, reduced amplitude of Δ_m today as the new physics of part (d). Explain briefly and qualitatively how measurements of large-scale structure could still distinguish this higher dark energy density scenario from the new physics model of part (d).

4

Consider a standard single-field slow-roll inflation model, where ϕ is the inflation field and $V(\phi)$ is its potential. You may assume throughout the entire problem that $a(\tau) = -(H\tau)^{-1}$ (with τ the conformal time) and that $H = \sqrt{\frac{V(\phi)}{3M_{\rm pl}^2}} \approx$ constant.

(a) Canonical quantization leads to the following expression for the field operator $\hat{f} = a\hat{\delta\phi}$, describing perturbations to the inflation field $\delta\phi$:

$$\hat{f}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left[f_{\mathbf{k}}(\tau)\hat{a}^{\dagger}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + f^*_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \right]$$

where $f^*_{\mathbf{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau})$ and $\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}$ are lowering and raising operators. State the commutation relations obeyed by $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}'}$. By calculating the two point correlation function of $\delta\phi$, deduce the dimensionless power spectrum of $\delta\phi$. Evaluate it when $k \ll aH$, and show that the spectrum is given by

$$\Delta^2_{\delta\phi} = \left(\frac{H}{2\pi}\right)^2 . \tag{1}$$

[*Hint: you may assume that the dimensionless power spectrum* $\Delta^2_{\delta\phi}$ *is related to the two point correlation function via* $\langle 0|\hat{\delta\phi}(\tau, \mathbf{x})\hat{\delta\phi}(\tau, \mathbf{x} + \mathbf{r})|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{2\pi^2}{k^3} \Delta^2_{\delta\phi} e^{-i\mathbf{k}\cdot\mathbf{r}}$]

(b) Briefly physically motivate the following relation between the comoving curvature perturbation $\mathcal{R}$ and $\delta\phi$: $\mathcal{R} = \frac{H}{\dot{\phi}}\delta\phi$. Deduce that the power spectrum of curvature perturbations is

$$\Delta^2_{\mathcal{R}}(k) = \frac{1}{2\epsilon \mathrm{Mpl}^2} \left(\frac{H}{2\pi}\right)^2 , \tag{2}$$

where $\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{(\dot{\phi})^2}{2H^2\mathrm{Mpl}^2}$ is the Hubble slow-roll parameter. Specify when the right hand side of this equation is to be evaluated.

(c) Show that in terms of derivatives of the inflationary potential, the scalar spectral index $n_s \equiv 1 + \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k}$ is given by

$$n_s - 1 = -3M_{\rm pl}^2 \left(\frac{V'}{V}\right)^2 + 2M_{\rm pl}^2 \left(\frac{V''}{V}\right) , \tag{3}$$

where primes indicate derivatives with respect to ϕ. [*Hint: you may assume that for the Hubble slow-roll parameters* $\epsilon = -\frac{d\ln H}{dN}$ *and* $\eta = \frac{d\ln\epsilon}{dN}$ *the following relations hold during slow-roll inflation:* $\frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 = \epsilon$ *and* $M_{\rm pl}^2\left(\frac{V''}{V}\right) = 2\epsilon - \frac{\eta}{2}$.]

(d) A far-future CMB experiment measures $n_s - 1 = -0.04$, with negligible error. The parameter r, the tensor-to-scalar ratio, is also measured to be $r = 0.001$ with negligible error; you may assume without proof that r is related to the slow-roll parameter by $r = 16\epsilon$. Assuming an inflation model described by a potential $V(\phi) = \lambda\phi^4$ for a constant λ, derive a relation between $n_s - 1$ and r. Is this inflation model consistent with the experimental measurements?

END OF PAPER