UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2019 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Starting from the inhomogeneous wave equation in flat space

$$\Box h_{ab} = -16\pi(T_{ab} - \frac{1}{2}\eta_{ab}T)$$

where h_{ab} is the metric perturbation, $\Box$ is the flat space wave operator, T_{ab} is the energy-momentum tensor of a source of gravitational radiation and $T = T_{ab}\eta^{ab}$, give a derivation of the quadrupole formula for the total time averaged power in gravitational radiation produced by the source

$$\frac{4\pi}{5}\dddot{Q}_{ij}\dddot{Q}_{ij}$$

where Q_{ij} is the mass quadrupole moment and a dot denotes the time derivative. The quadrupole moment is defined by

$$Q_{ij} = \int \rho(x_i x_j - \frac{1}{3}\delta_{ij}x^2)\,dV$$

where ρ is the mass density in the radiating body.

A planet of mass M orbits a star in a circular orbit of radius R and period τ. Find the total power radiated in gravitational radiation by this system.

2

A spacetime has metric g_{ab}. Describe how to construct an orthonormal basis of one-forms from the metric.

Describe the construction of the connection 1-form, the curvature 2-form and the Ricci tensor.

Consider the metric

$$ds^2 = -dt^2 + e^{-2t}(dx^2 + dy^2 + dz^2).$$

Calculate the connection 1-form.

Calculate the curvature 2-form.

Show that the vacuum Einstein equations are satisfied with a non-vanishing cosmological constant.

What is the value of the cosmological constant?

3

Anti-de Sitter space has metric

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where θ and ϕ are the polar angles on a sphere, $0 < r < \infty$ and $-\infty < t < \infty$.

Write down an action principle for finding the geodesics in this spacetime.

Find the geodesic equations.

Carefully explain why spherical symmetry allows one, without any loss of generality, to choose to examine only those geodesics that always lie in the equatorial plane $\theta = \frac{\pi}{2}$. Draw a diagram illustrating your answer.

Suppose that

$$E = (1+r^2)\dot{t}$$

and

$$L = r^2\dot{\phi}.$$

Explain why E and L are constants.

Find a first order equation for the radial motion of a timelike particle.

Suppose a particle starts at $r = 0$, and moves away from $r = 0$. Find the proper time for this particle that has elapsed when it first returns to $r = 0$.

4

The Riemann tensor is defined in terms of the Christoffel symbols by

$$R^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^f_{bd}\Gamma^a_{fc} - \Gamma^f_{bc}\Gamma^a_{fd}.$$

Derive the Bianchi identities.

Derive the contracted Bianchi identities.

The action for general relativity coupled to a massive vector field A_a is

$$I = \frac{1}{16\pi}\int (R - 2\Lambda)\sqrt{-g}d^4x + \int (-\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}m^2 A_a A^a)\sqrt{-g}d^4x.$$

where R is the Ricci scalar, Λ is the cosmological constant and the field strength of the vector field, F_{ab} is given by $F_{ab} = \nabla_a A_b - \nabla_b A_a$.

Suppose the metric is varied by $g_{ab} \to g_{ab} + h_{ab}$. The Ricci scalar then also varies as $R(g+h) = R(g) - \nabla_a \nabla^a h + \nabla_a \nabla_b h^{ab} - R_{ab}h^{ab}$ to linear order in h.

By varying the metric find the Einstein equations for the combined gravitational and massive vector field system, $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} + m^2 A_a A_b - \frac{1}{2}m^2 g_{ab}A_c A^c)$.

By varying A_a, find the equation of motion for the massive vector field.

Show that $\nabla_a A^a = 0$.

Show that $\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$.

By using the contracted Bianchi identities, show that the equation of motion for A_a is consistent with the Einstein equations.

END OF PAPER