UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2019 1:30 pm to 3:30 pm

PAPER 308

CLASSICAL AND QUANTUM SOLITONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A scalar field theory in one spatial dimension has Lagrangian

$$L = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - \frac{1}{2}(q^2 - \phi^2)^2 (p^2 - \phi^2)^2 \right\} dx$$

where $q > p > 0$. Show that there are three types of kink and three types of antikink in this theory. Using the appropriate Bogomolny equations, determine the kink energies, and the kink solution for the kink that passes through $\phi = 0$.

Find to leading order in ε the ratios of the kink energies when $q = p + \varepsilon$, for ε small and positive. Find the kink solution in the case that $q = p$.

2 Let Σ be a surface with local complex coordinate z and metric $ds^2 = \Omega(z, \bar{z})\, dz d\bar{z}$. The Bogomolny equations for Abelian Higgs vortices on Σ are

$$\partial_{\bar{z}}\phi - i a_{\bar{z}}\phi = 0\,, \tag{1}$$

$$-2i f_{z\bar{z}} - \frac{\Omega}{2}(1 - |\phi|^2) = 0\,, \tag{2}$$

where $f_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z$. Show that equations (1) and (2) imply that (away from zeros of ϕ)

$$-\frac{2}{\Omega}\nabla^2 \log|\phi| = 1 - |\phi|^2\,, \tag{3}$$

where $\nabla^2 = 4\partial_z\partial_{\bar{z}}$.

The Gaussian curvature of Σ is

$$K = -\frac{1}{2\Omega}\nabla^2 \log\Omega\,.$$

Consider now a general Higgs field ϕ and define a new metric $\widetilde{ds}^2 = \widetilde{\Omega}\, dz d\bar{z}$ on Σ, where $\widetilde{\Omega} = \Omega|\phi|^2$. Let the new Gaussian curvature be $\widetilde{K}$. Show that if

$$(\widetilde{K} + \tfrac{1}{2})\widetilde{\Omega} = (K + \tfrac{1}{2})\Omega$$

then the vortex equation (3) is satisfied. Show that the metric

$$ds^2 = \frac{8n^2|z|^{2n-2}}{(1 - |z|^{2n})^2}\, dz d\bar{z}\,, \tag{4}$$

for $0 < |z| < 1$, has Gaussian curvature $K = -\frac{1}{2}$ for all positive integers n. By considering a ratio of these metrics for distinct values of n, find a vortex solution with winding number N on the hyperbolic plane. [You should find $|\phi|$, and then make a gauge choice to determine $\phi, a_{\bar{z}}$ and a_z. The metric on the hyperbolic plane (in the Poincaré disc model) is the case $n = 1$ of (4), and it extends smoothly to $z = 0$.]

3 Define the algebraic degree and the Wronskian of a rational map between 2-spheres

$$R(z) = \frac{p(z)}{q(z)}\,,$$

and determine these for

$$\text{(i)}\quad R(z) = z^2\,,$$

$$\text{(ii)}\quad R(z) = \frac{\sqrt{3}iz^2 - 1}{z^3 - \sqrt{3}iz}\,.$$

Determine the rotational symmetries of the map $R(z) = z^2$, identifying the axes and angles of rotation.

Explain how rational maps can be used to construct (approximate) Skyrmions, and how the algebraic degree and the Wronskian affect the properties of the Skyrmion.

Briefly describe the Skyrmion obtained using the rational map $R(z) = z^2$, and explain how this Skyrmion can be quantized. Determine all the allowed quantum states with both spin and isospin no greater than 1.

END OF PAPER