UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 1:30 pm to 3:30 pm

PAPER 307

SUPERSYMMETRY

Attempt Question 1 and then EITHER Question 2 OR Question 3.

Question 1 carries 20 marks. Questions 2 and 3 each carry 30 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $(x^\pm;\theta^\pm,\bar\theta^\pm)$ be coordinates on $\mathbb{R}^{2|4}$ and let

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\frac{\partial}{\partial x^\pm} \qquad \text{and} \qquad \bar{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\frac{\partial}{\partial x^\pm}\,.$$

Define the *chiral derivatives* $D_\pm$ and $\bar D_\pm$, and show that $\{D_+,\mathcal{Q}_\pm\}=0$ and $\{D_+,\bar{\mathcal{Q}}_\pm\}=0$.

A *twisted* chiral superfield U obeys $D_-U=0$ and $\bar D_+U=0$. Give the component expansion of U in terms of appropriate twisted chiral coordinates on $\mathbb{R}^{2|2}$.

Write down the most general form of a supersymmetric action in a theory of a single chiral superfield Φ and a single twisted chiral superfield U, together with their conjugates. Explain why the terms you have written down are indeed supersymmetric. [*You are not required to give the action in component field form.*]

Define how axial and vector $U(1)$ transformations act on Φ and U, and give conditions under which your action is invariant under each of these transformations at the classical level.

2

a) For $a=1,\ldots,n$, let $\eta^a(t)$ be complex fermions and $\bar\eta_a(t)$ their Hermitian conjugates. Consider the worldline Euclidean action

$$S[\eta,\bar\eta] = \int_0^1 \bar\eta_a\dot\eta^a + \bar\eta_a T^a{}_b\eta^b\,dt$$

where T is a constant, diagonalizable, traceless, $n\times n$ matrix. Find all canonical commutation relations between the corresponding operators $\hat\eta^a$ and $\hat{\bar\eta}_b$.

b) Identify the Hilbert space $\mathcal{H}$ of this model. Let $N=\hat\eta^a\hat{\bar\eta}_a$. How does $(-1)^N$ act on a general state $\Psi\in\mathcal{H}$?

c) Briefly explain why $\mathrm{Tr}_\mathcal{H}((-1)^N\,e^{-\hat{\bar\eta}_aT^a{}_b\hat\eta^b})$ can be written as a fermionic path integral

$$\int e^{-S[\eta,\bar\eta]}\,D\eta\,D\bar\eta\,,$$

where $\eta^a(0)=\eta^a(1)$ and $\bar\eta_a(0)=\bar\eta_a(1)$.

d) By computing the path integral, interpreted as an integral over all the Fourier modes of the fields, show that

$$\mathrm{Tr}_\mathcal{H}((-1)^N\,e^{-\hat{\bar\eta}_aT^a{}_b\hat\eta^b}) = \det(1-e^{-T})\,.$$

[*You may assume that* $\prod_{k=1}^\infty(2\pi k)^2=1$ *in* ζ*-function regularization.*]

3

Consider supersymmetric quantum mechanics with action

$$S[X] = \frac{i}{2} \int_{\mathbb{R}^{1,1}} g_{ab}(X) \, \frac{dX^a}{dt} \, DX^b \; d\theta \, dt \, .$$

Here $X^a(t,\theta) = x^a(t) + \theta\psi^a(t)$ is a superfield describing maps $X : \mathbb{R}^{1,1} \to M$, where M is an even dimensional manifold with Riemannian metric g_{ab}, while $D = \partial/\partial\theta - i\theta \, \partial/\partial t$.

a) Obtain the component form of the action, showing that it is invariant under diffeomorphisms of M.

b) Show further that, provided the fields decay appropriately as $|t| \to \infty$, the action is invariant under the transformations

$$\delta x^a = \epsilon\psi^a \, , \qquad \delta\psi^a = -i\epsilon \frac{dx^a}{dt}$$

where ϵ is a constant fermionic parameter. Find the corresponding Noether charge Q.

c) Explain how the Hilbert space $\mathcal{H}$ of the corresponding quantum theory can be understood in terms of the geometrical objects living on M.

Now let $f : M \to M$ be an isometry of g and consider the modified trace $\mathrm{Tr}_{\mathcal{H}}((-1)^F f \, e^{-\beta H})$, where H is the Hamiltonian of the theory and F the fermion number operator.

d) Express this modified trace in terms of a path integral, stating the periodicity conditions imposed on the fields $x^a(t)$ and $\psi^a(t)$.

e) Explain why your path integral localizes to a neighbourhood of the set $M^f \subset M$ that is fixed by f.

END OF PAPER