UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 1:30 pm to 4:30 pm

PAPER 306

STRING THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Given the stress tensor for the Bosonic string

$$T(z) = -\frac{1}{\alpha'} : \partial X^{\mu}(z) \partial X_{\mu}(z) :,$$

find the operator product expansion for $T(z)T(\omega)$ and hence show that under infinitesimal conformal transformations where $z \to z + v(z)$, $T(z)$ transforms as

$$\delta_v T(z) = \frac{c}{12} \partial^3 v(z) + 2\partial v(z)\, T(z) + v(z) \partial T(z),$$

where c is a constant you should find.

Show that the modes L_n of $T(z)$ satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0},$$

where you should find an expression for $A(m)$. What is the physical significance of $A(m)$?

2 Starting with the path integral expression

$$\left\langle e^{-\int_{\Sigma} \mathrm{d}^2 z J_{\mu} X^{\mu}} \right\rangle = \int \mathcal{D}X \, \exp\left(-\frac{1}{2\pi\alpha'} \int_{\Sigma} \mathrm{d}^2 z \, \eta_{\mu\nu} \partial X^{\mu} \bar{\partial} X^{\nu} - \int_{\Sigma} \mathrm{d}^2 z \, J_{\mu} X^{\mu} \right)$$

show that

$$\left\langle e^{-\int_{\Sigma} \mathrm{d}^2 z J_{\mu} X^{\mu}} \right\rangle \propto \exp\left(\frac{1}{2} \int_{\Sigma \times \Sigma} \mathrm{d}^2 z \mathrm{d}^2 \omega J(z) G(z,\omega) J(\omega) \right) \times \int \mathrm{d}^D x \, \exp\left(-x^{\mu} \int_{\Sigma} \mathrm{d}^2 z J_{\mu} \right) \tag{1}$$

where D is the dimension of spacetime, x^{μ} is that part of the field X^{μ} which is independent of z and $\bar{z}$ and $G(z,\omega)$ satisfies

$$-\frac{1}{\pi\alpha'} \partial \bar{\partial} G(z,\omega) = \delta^2(z-\omega).$$

Using

$$G(z,\omega) = -\frac{\alpha'}{2} \ln |z-\omega|^2,$$

and an appropriate choice for $J_{\mu}(z,\bar{z})$ in (1) show that

$$\left\langle e^{ik_1 \cdot X(z_1,\bar{z}_1)} \, e^{ik_2 \cdot X(z_2,\bar{z}_2)} ... e^{ik_n \cdot X(z_n,\bar{z}_n)} \right\rangle \propto \delta^D \left(\sum_{j=1}^{n} k_{j\mu} \right) \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j}.$$

Hence derive an expression for the tree-level scattering amplitude for three tachyons, each of mass $m^2 = -4/\alpha'$, explaining the origin of the various contributions to the amplitude.

3 With reference to the gauge-fixing fermion Ψ, explain why the BRST operator Q_B is required to satisfy $\{Q_B, Q_B\} = 0$. Hence show that the action

$$S = S_0 + \{Q_B, \Psi\},$$

is BRST-invariant if S_0 is invariant under a particular gauge symmetry and Ψ is the gauge-fixing fermion associated with this gauge symmetry.

The field $\phi(z)$ has conformal weight $(h, \bar{h}) = (1, 0)$. By considering a suitable expansion for $\phi(z)$ in terms of modes ϕ_n, briefly explain why

$$\phi_n|0\rangle = 0, \qquad n > -h,$$

where $|0\rangle$ is the vacuum state. Hence, by considering the mode expansion of the $b(z)$ and $c(\omega)$ ghosts, derive the operator product expansion for $b(z)c(\omega)$. You may assume that $|z| > |\omega|$.

The ghost stress tensor is given by

$$T_{\text{gh}}(z) = \partial b(z)\, c(z) - 2\partial(b(z)c(z)).$$

Derive the BRST transformation of the ghost field $c(z)$ and hence, given a field $\Phi(z, \bar{z})$ of weight $(1, 1)$, write down two types of vertex operator (one local, the other integrated over the worldsheet) and show that they are BRST-invariant.

4 The stress tensor is

$$T(z) = -\frac{1}{\alpha'} : \partial X^{\mu}(z) \partial X_{\mu}(z) :$$

Find an expression for the mode operators L_n of the stress tensor in terms of the modes α^{μ}_{n} of the embedding fields X^{μ}. You may use the mode expansion

$$\partial X^{\mu}(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n} \alpha^{\mu}_{n} z^{-n-1}.$$

Derive an expression for the operator product expansion $X^{\mu}(z,\bar{z})X^{\nu}(\omega,\bar{\omega})$ and hence find an expression for the commutator $[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}]$.

Explain the relationship between the constraints arising from the vanishing of the stress tensor and the mass-shell condition in spacetime and find expressions for the masses of the states

$$|T\rangle = |k\rangle, \qquad |g\rangle = \varepsilon_{\mu\nu}\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|k\rangle,$$

where k is a spacetime momentum and $\varepsilon_{\mu\nu}$ is a constant polarisation tensor. What other conditions must $|g\rangle$ satisfy in order to be physical?

Explain the relationship between the state $|g\rangle$, the operator

$$\varepsilon_{\mu\nu}\partial X^{\mu}(z)\bar{\partial}X^{\nu}(\bar{z})\, e^{ik\cdot X(z,\bar{z})},$$

and deformations of the background spacetime metric.

END OF PAPER