UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2019 9:00 am to 12:00 pm

PAPER 305

THE STANDARD MODEL

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Recall the expansion for a Dirac field $\psi(x)$ with mass m,

$$\psi(x) = \sum_{p,s} \left[b^s(p) u^s(p) e^{-ip\cdot x} + d^{s\dagger}(p) v^s(p) e^{ip\cdot x} \right] ,$$

where $(\not{p} - m)u^s(p) = 0$, $(\not{p} + m)v^s(p) = 0$ and $s = \pm\frac{1}{2}$.

(a) Briefly explain the meaning of b^s, $d^{s\dagger}$, u^s and v^s.

(b) Assuming the results,

$$u^s(p_P) = \gamma^0 u^s(p) , \qquad v^s(p_P) = -\gamma^0 v^s(p) ,$$

$$\hat{P} b^s(p) \hat{P}^{-1} = \eta_P b^s(p_P) , \qquad \hat{P} d^{s\dagger}(p) \hat{P}^{-1} = -\eta_P d^{s\dagger}(p_P) ,$$

show that under a parity transformation (P),

$$\psi(x) \mapsto \hat{P}\psi(x)\hat{P}^{-1} = \eta_P \gamma^0 \psi(x_P) ,$$

where $|\eta_P| = 1$, $x_P^\mu = (x^0, -\mathbf{x})$ and $p_P^\mu = (p^0, -\mathbf{p})$.

(c) Assuming that $\psi(x)$ satisfies the Dirac equation, show that $\psi^P(x) \equiv \hat{P}\psi(x)\hat{P}^{-1}$ also satisfies the Dirac equation.

For the remainder of this question, you may assume that $\hat{P}\bar{\psi}(x)\hat{P}^{-1} = \eta_P^* \bar{\psi}(x_P)\gamma^0$ and under a charge-conjugation transformation (C), $\psi(x) \mapsto \hat{C}\psi(x)\hat{C}^{-1} = C\bar{\psi}^T(x)$ and $\hat{C}\bar{\psi}(x)\hat{C}^{-1} = -\psi^T(x)C^{-1}$, where $\gamma^{\mu\,T} = -C^{-1}\gamma^\mu C$.

(d) Given that the interaction between a photon and an electron is parity invariant and charge-conjugation invariant, derive an expression for the P and C transformed photon fields, $\hat{P}A_\mu(x)\hat{P}^{-1}$ and $\hat{C}A_\mu(x)\hat{C}^{-1}$.

(e) How does $ia\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu}$ transform under P, C and the combination CP? Here a is a real constant, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. What can you infer about the transformation properties of this term under time-reversal? Could such an interaction arise in the Standard Model?

2

(a) Consider a field theory with an n-component real scalar field ϕ and a potential $V(\phi)$. The potential is invariant under global infinitesimal transformations of the field,

$$\phi \mapsto \phi + i\alpha^a t^a \phi\,,$$

where t^a are the generators corresponding to the group $G = SU(N)$ and α^a are infinitesimal parameters. The potential is minimized by $\phi \in \Phi_0 = \{\phi_0 \,|\, V(\phi_0) = V_{\min}\}$ and a given vacuum ϕ_0 is invariant under transformations belonging to the normal subgroup H, i.e.,

$$\tilde{t}^i \phi_0 = 0\,,$$

where $\tilde{t}^i$ is a generator for H. By expanding $V(\phi)$ about ϕ_0, prove that there are dim G − dim H massless scalar modes. [You may consider the theory at a classical level and ignore quantum corrections.]

(b) Now consider an $SU(2)$ gauge theory involving a 2-component complex scalar field ϕ_i $(i = 1, 2)$ and three gauge fields B^a_μ $(a = 1, 2, 3)$ with Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2\,, \quad m^2 < 0\,, \quad \lambda > 0\,,$$

where $D_\mu \phi = \big(\partial_\mu + igB^a_\mu \tau^a\big)\phi$, $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - g\epsilon^{abc} B^b_\mu B^c_\nu$, $[\tau^a, \tau^b] = i\epsilon^{abc}\tau^c$ and the generators are related to the Pauli σ matrices by $\tau^a = \sigma^a/2$.

Discuss how the symmetry is spontaneously broken by the vacuum. Why, without loss of generality, can we take the vacuum to be $\phi_0 = \frac{1}{\sqrt{2}}(0, v)^T$ and the fluctuations of ϕ about the vacuum to be $\phi(x) = \frac{1}{\sqrt{2}}(0, v + h(x))^T$, where v and $h(x)$ are real? Write the Lagrangian density in terms of the physical fields and give their masses (ignoring any quantum corrections). Draw Feynman diagrams for all their tree-level interactions. [It is not necessary to derive the Feynman rules.] Comment on the number of massless fields based on symmetry principles.

3

(a) State whether or not each of the following processes is allowed at tree level in the Standard Model. For those which are allowed, draw all possible tree-level Feynman diagrams.

(i) $u\,\bar{c} \to c\,\bar{u}$ (ii) $u\,\bar{c} \to d\,\bar{s}$ (iii) $\nu_e\mu^- \to \bar{\nu}_e\mu^+$ (iv) $\nu_e e^+ \to \nu_e e^+$ (v) $\nu_e e^- \to \nu_e e^-$.

(b) For the remainder of this question, treat neutrinos and electrons as massless, neglect mixing between different generations of leptons, and assume that there are no right-handed neutrinos. The relevant part of the effective weak Lagrangian density for $\nu_e e^- \to \nu_e e^-$ is,

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}}\left[\bar{\nu}_e\gamma^\alpha(1-\gamma^5)e\,\bar{e}\gamma_\alpha(1-\gamma^5)\nu_e + \bar{\nu}_e\gamma^\alpha(1-\gamma^5)\nu_e\,\bar{e}\gamma_\alpha(c_V - c_A\gamma^5)e\right] ,$$

where c_V and c_A are real constants. Explain how each term is related to the relevant Feynman diagram(s) you drew for (v) in part (a).

(c) Using $\mathcal{L}_{\text{eff}}$, show that the unpolarised cross section for $\nu_e(k)e^-(p) \to \nu_e(k')e^-(p')$ is,

$$\sigma(\nu_e e^- \to \nu_e e^-) = G_F^2\, H(s)\left(c_V^2 + Bc_A^2 + Cc_Vc_A + Dc_V + Ec_A + F\right) ,$$

where $H(s)$ is a function of $s = (p+k)^2$ and B, C, D, E and F are constants which you should find. [*Hints: Work in the centre of momentum frame. You may find it helpful to use the following Fierz identity:*

$$\left[\gamma^\alpha(1-\gamma^5)\right]_{ij}\left[\gamma_\alpha(1-\gamma^5)\right]_{kl} = -\left[\gamma^\alpha(1-\gamma^5)\right]_{il}\left[\gamma_\alpha(1-\gamma^5)\right]_{kj} .$$

The following expressions may be used without proof:

$$\begin{aligned}
\text{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) &= 0 \quad \textit{for } n \textit{ odd}, \\
\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \\
\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= -4i\epsilon^{\mu\nu\rho\sigma}, \\
\epsilon^{\alpha\beta\sigma\rho}\epsilon_{\alpha\beta\lambda\tau} &= -2(\delta^\sigma_{\ \lambda}\,\delta^\rho_{\ \tau} - \delta^\sigma_{\ \tau}\,\delta^\rho_{\ \lambda}),
\end{aligned}$$

and the differential cross section for $A(p_A) + B(p_B) \to C(p_C) + D(p_D)$ is,

$$d\sigma = \frac{1}{|\vec{v}_A - \vec{v}_B|}\frac{1}{4p_A^0p_B^0}\left(\frac{d^3p_C}{(2\pi)^3 2p_C^0}\right)\left(\frac{d^3p_D}{(2\pi)^3 2p_D^0}\right)(2\pi)^4\delta^{(4)}(p_A + p_B - p_C - p_D)\,|\mathcal{M}|^2 .]$$

(d) Comment on the behaviour of the cross section for large s.

4

To leading order, the scale dependence of a renormalized coupling $g_i(\mu)$ is given by

$$\mu \frac{dg_i}{d\mu} = -\frac{\beta_i}{16\pi^2} g_i^3 \,,$$

where μ is the renormalization scale and β_i is a real constant. Let α_1, α_2 and α_3 be the coupling constants of the Standard Model gauge groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ respectively, where $\alpha_i = g_i^2/4\pi$.

(a) Briefly explain the consequences of β_i being positive (as in QCD) or negative.

(b) Derive an expression for $\alpha_3(\mu)$ in terms of an energy scale Λ where α_3 diverges.

(c) Find an expression relating $\alpha_i(m_Z)$ to $\alpha_i(\mu)$, where m_Z is the mass of the Z boson.

(d) Suppose there exists a scale, $M_{\rm GUT} > m_Z$ for which

$$\frac{5}{3}\alpha_1(M_{\rm GUT}) = \alpha_2(M_{\rm GUT}) = \alpha_3(M_{\rm GUT}) \,.$$

Show that this implies,

$$\alpha_3^{-1}(m_Z) = \alpha_2^{-1}(m_Z) + \frac{\beta_3 - \beta_2}{\frac{3}{5}\beta_1 - \beta_2}\left[\frac{3}{5}\alpha_1^{-1}(m_Z) - \alpha_2^{-1}(m_Z)\right] .$$

(e) For a gauge group $SU(N)$ which is coupled to n_L left-handed fermions, n_R right-handed fermions and n_s complex scalars, all in the fundamental representation,

$$\beta_i = \frac{11}{3}N - \frac{1}{3}(n_L + n_R) - \frac{1}{6}n_s \,.$$

Assuming that there are no right-handed neutrinos, list the scalar and first-generation fermion fields in the Standard Model and give their $SU(2)_L$ and $SU(3)_C$ representations. Hence, calculate β_2 and β_3 for the Standard Model at an energy scale above the mass of the top quark.

END OF PAPER