MAT3, MAMA, NST3PHY, MAPY

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2019 9:00 am to 12:00 am

PAPER 304

ADVANCED QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider the classical action $S[\phi]$ for a scalar field in d dimensions. Write down a generating functional $Z[J]$ for quantum expectation values, along with an example of how it is used.

(b) In a few sentences and equations, explain the relation between the action $S[\phi]$, the Wilsonian effective action $W[J]$, and the quantum effective action $\Gamma[\Phi]$, including definitions of J and Φ.

(c) Given that $Z[J]$ may be expressed in a perturbative expansion as a sum of Feynman diagrams, how can $W[J]$ and $\Gamma[\Phi]$ be expressed in perturbative expansions? Justify your answers mathematically.

(d) Consider a real scalar field in 0-dimensions (i.e. ϕ is a real number) with action $S(\phi)$. Writing $\phi = \phi_0 + \eta$, we define an effective action for a particular ϕ_0 as

$$W(J;\phi_0) \equiv -\hbar \log \int d\eta \, \exp\left[-\frac{1}{\hbar}\left(S(\phi_0+\eta) + J\eta\right)\right] .$$

Given some field χ, define J_χ through

$$\left.\frac{\partial W(J;\phi_0)}{\partial J}\right|_{J=J_\chi} = \chi \, .$$

Finally let $\Gamma(\chi;\phi_0)$ be the Legendre transform of $W(J_\chi;\phi_0)$. Working *nonperturbatively* show that

$$\Gamma(\eta;\phi_0) = \Gamma(\phi_0+\eta;0)$$

where $\Gamma(\phi_0+\eta;0) = \Gamma(\phi)$.

2

Consider the following interacting scalar field theory in 4 Euclidean dimensions with a momentum cutoff Λ_0:

$$S_{\Lambda_0}[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m_0^2\phi^2 + \frac{h_0}{3!}\phi^3 + \frac{g_0}{4!}\phi^4 \right) .$$

The partition function is written as

$$\mathcal{Z}_{\Lambda_0}(m_0^2, h_0, g_0) = \int^{\Lambda_0} \mathcal{D}\phi\, e^{-S_{\Lambda_0}[\phi]}$$

where this should be interpreted as a functional integral over momentum modes $\tilde{\phi}(p)$ of ϕ with momentum satisfying $p^2 \leqslant \Lambda_0^2$.

(a) How is the effective action $S_\Lambda^{\text{eff}}[\phi]$ at a lower scale $\Lambda \leqslant \Lambda_0$ defined? Show that

$$S_\Lambda^{\text{eff}}[\phi] = S_{\Lambda_0}[\phi] - \log\left[\int_\Lambda^{\Lambda_0} \mathcal{D}\phi^+ \,\exp\left(-\Delta S[\phi,\phi^+]\right)\right]$$

where the meaning of ϕ^+ should be explained. Give an expression for $\Delta S[\phi,\phi^+]$ for cases (i) $h_0 \neq 0$ and (ii) $h_0 = 0$.

(b) By expressing a generic effective action as a series of terms, each of dimension d_i and consisting of n_i fields,

$$S_\Lambda^{\text{eff}}[\phi] = \int d^4x \left[\frac{Z_\Lambda}{2}\left(\partial_\mu\phi\,\partial^\mu\phi + m^2(\Lambda)\phi^2\right) + \sum_i \frac{Z_\Lambda^{n_i/2}}{\Lambda^{d_i-4}} g_i(\Lambda) O_i(x)\right]$$

obtain the Callan-Symanzik equation governing the Λ-dependence of these couplings.

(c) Obtain a Callan-Symanzik equation for the 4-point function $\Gamma_\Lambda^{(4)}\big(x_1,x_2,x_3,x_4; m^2, g_i(\Lambda)\big)$.

(d) In a few paragraphs, explain what is meant by "the continuum limit" and possible behaviours theories may exhibit in this limit.

3

The Feynman-gauge QED action in d Euclidean spacetime dimensions is $S = \int d^d x \, \mathcal{L}$ where

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2}(\partial_\mu A)(\partial^\mu A)$$

where $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Let $S_F(\not{p}) = (i\not{p} + m)^{-1}$ be the free fermion propagator and $G(\not{p})$ be the exact fermion propagator

$$G(\not{p}) = \int d^4x \, e^{-ip\cdot(x-y)} \langle \psi(x)\bar{\psi}(y) \rangle \, .$$

Denote by $\Sigma(\not{p})$ the fermion self-energy, that is, the sum of one-particle irreducible diagrams with 2 external, amputated fermion legs. How can $G(\not{p})$ be written in terms of $S_F(\not{p})$ and $\Sigma(\not{p})$? What is the relationship between the exact propagator and the physical fermion mass m_{phys}?

(b) Draw the one-loop Feynman diagram which contributes to $\Sigma(\not{p})$. With just a few words and no lengthy mathematical derivations, explain why this contribution can be written in 4 Euclidean dimensions as follows

$$\Sigma(\not{p}) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \, \gamma^\mu \, \frac{-i\not{k} + m}{k^2 + m^2} \, \gamma_\mu \, \frac{1}{(p-k)^2} \, .$$

(c) Working in $d = 4 - \epsilon$ dimensions, discuss the dimensions carried by the coupling e (in units where $\hbar = c = 1$).

[QUESTION CONTINUES ON THE NEXT PAGE]

(d) Show that the integral above can, in d dimensions, be expressed as

$$\Sigma(\not{p}) = -\frac{e^2}{(4\pi)^{d/2}}\,\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\,\frac{C\not{p}+Fm}{\Delta^{\epsilon/2}}$$

where you should determine C, F and Δ. C and F depend on x and ϵ, and Δ depends on x, p^2, and m^2. *[Hint: Use the identities at the end of the question. Ignore the divergence as* $k \to 0$.*]*

(e) Expand $\Sigma(\not{p})$ about $\epsilon = 0$ in order to identify the divergent and finite terms. In either the minimal subtraction (MS) or modified minimal subtraction ($\overline{\text{MS}}$) schemes write down the 1-loop contribution to the renormalized mass and relate the renormalized mass m to the physical fermion mass m_{phys}. *[Hint: Use your answer to Part (b). You do not need to evaluate any remaining integral over a Feynman parameter* x.*]*

[Useful identities:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA+(1-x)B]^2}\,.$$

In d dimensions

$$\gamma^\mu\gamma_\mu = d$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu\,.$$

For appropriate integrands, one can perform the angular integrals by directly replacing

$$\frac{d^d\ell}{(2\pi)^d} \to \frac{(\ell^2)^{\frac{d}{2}-1}\,d\ell^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\,.$$

The following integral may be used without proof

$$\int_0^\infty \frac{(\ell^2)^{\frac{d}{2}-1}\,d\ell^2}{(\ell^2+\Delta)^2} = \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)}\,.$$

The expansion of the Γ function near zero is

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon)$$

where γ is the Euler-Mascheroni constant.]

4

Consider an SU(N) gauge theory with Lagrangian

$$S_g = \int d^4x \, \frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, and f^{abc} are the structure constants appearing in the Lie algebra, $[t^a, t^b] = i f^{abc} t^c$.

(a) Define a Wilson line $U(y,x)$ to be an SU(N) element depending on two spacetime points with the properties: (i) Under a gauge transformation $V(x) \in \mathrm{SU}(N)$, $U(y,x) \mapsto V(y) U(y,x) V^\dagger(x)$; (ii) For small a and a unit vector n, $U(x + an, x) = 1 + igan^\mu A^a_\mu t^a + O(a^2)$. Show that under an infinitesimal gauge transformation, $V(x) = 1 + i\alpha^a(x) t^a$,

$$A^a_\mu \mapsto (A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c \, .$$

(b) Given a gauge-fixing condition $G[A] = 0$, use the identity

$$1 = \int \mathcal{D}\alpha(x) \, \delta(G[A^\alpha]) \, \det\left(\frac{\delta G[A^\alpha]}{\delta \alpha} \right)$$

to show that, for Lorenz gauge $\partial^\mu A^a_\mu = 0$, the action can be written as

$$S = S_g + \int d^4x \left[\frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 + \bar{c}\, \partial^\mu D_\mu c \right] ,$$

where c and $\bar{c}$ are anticommuting ghost fields and you should find an expression for D_μ.

(c) Using the action from Part (b), show that the Feynman rule for the momentum-space gauge boson propagator is equal to

$$\frac{1}{k^2} \left(X \delta^{\mu\nu} + Y \frac{k^\mu k^\nu}{k^2} \right)$$

where you should determine X and Y.

(d) In a few sentences, explain (i) Why gauge-fixing is necessary for path-integral quantization of gauge fields; (ii) What role is played by the fields c and $\bar{c}$ above.

(e) Show that it is not necessary to introduce ghosts for the gauge-fixing condition $n \cdot A^a$, where n is a constant unit vector.

END OF PAPER