UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 9:00 am to 11:00 am

PAPER 303

STATISTICAL FIELD THEORY

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Briefly describe the three steps of the renormalisation group (RG) and explain how they result in a flow of the parameters that specify the free energy.

Consider the free energy for a real scalar field ϕ with cubic interaction,

$$F[\phi] = \int d^dx \; \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + \lambda_0\phi^3 .$$

Perform the first step of RG to determine the shifted coupling ${\mu'}^2$ to leading, non-trivial order in λ_0.

Perform the first step of RG to determine the shifted coupling λ' to leading, non-trivial order in λ_0.

[*You may use Wick's theorem without proof, and the fact that, in the absence of interactions,* $\langle\phi_{\mathbf{k}}\phi_{\mathbf{k}'}\rangle = (2\pi)^d\delta^d(\mathbf{k}+\mathbf{k}')G_0(k)$ *where* $G_0(k) = (k^2+\mu_0^2)^{-1}$. *You may leave your final answer for* ${\mu'}^2$ *and* λ' *in integral form.*]

2

A system is described by two, real scalar fields ϕ_1 and ϕ_2. The free energy is given by

$$F[\phi_1,\phi_2] = \int d^dx \; \frac{1}{2}(\nabla\phi_1)^2 + \frac{1}{2}(\nabla\phi_2)^2 + \frac{\mu_1^2}{2}\phi_1^2 + \frac{\mu_2^2}{2}\phi_2^2 + g(\phi_1^2+\phi_2^2)^2$$

with $g > 0$, while μ_i^2, $i = 1,2$ can take either sign.

a) Suppose that $|\mu_1^2| > |\mu_2^2|$. Find the mean field ground state of the system for different values of μ_i^2, $i = 1,2$ and describe the different symmetry breaking patterns that occur. Determine the critical exponents β_i, defined as $\phi_i \sim (-\mu_i)^{2\beta_i}$, for $i = 1,2$ close to the critical point.

What is meant by the *lower critical dimension.* Identify the lowest critical dimension for the system above and give an argument that no ordered phase exists. [You need not solve for the relevant field configurations, but you should describe their properties.]

b) Suppose now that $\mu^2 \equiv \mu_1^2 = \mu_2^2$. What is the lower critical dimension? In the lower critical dimension, compute the anomalous dimension η of the complex scalar $\psi = \phi_1 + i\phi_2$ in the ordered phase. Give an argument that suggests a phase transition will occur when $\mu^2 = -4g/\pi$.

3

Briefly describe how one can assign scaling dimensions to interaction terms at a critical point. Explain how this can be used to separate interactions into relevant, irrelevant or marginal, and comment on the implication for universality.

At a critical point in d dimensions, the correlation function of a scalar field behaves as

$$\langle \phi(\mathbf{x})\phi(0)\rangle \sim \frac{1}{r^{d-2+\eta}}$$

for some critical exponent η. Close to this critical point, the correlation length ξ depends on the reduced temperature $t = |T - T_c|/T_c$ as $\xi \sim t^{-\nu}$ for some critical exponent ν.

i) Show that the heat capacity scales as $c \sim t^{-\alpha}$ with $\alpha = 2 - \nu d$.

ii) Show that, in the ordered phase, $\phi \sim t^{\beta}$ for some β that you should determine.

The beta functions, for $\mu^2 \sim t$ and a coupling g, computed perturbatively in g, are given by

$$\begin{aligned} \frac{d\mu^2}{ds} &= 2\mu^2 - \frac{2\Lambda^4}{\Lambda^2+\mu^2} g^2 + \mathcal{O}(g^6) \\ \frac{dg}{ds} &= bg^2 - cg^4 + \mathcal{O}(g^6) \end{aligned}$$

where $b, c > 0$ and Λ is the UV cut-off.

iii) Identify the Gaussian fixed point. What are the scaling dimensions of μ^2 and g about the Gaussian fixed point? What is the critical exponent ν about this fixed point?

iv) Identify a second, interacting fixed point. What is the requirement on b and c for us to trust the existence of this fixed point? Draw the RG flows for this system. What is the critical exponent ν at the interacting fixed point, computed to leading non-trivial order in g^2?

END OF PAPER