MAT3, MAMA, NST3AS

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2019 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the definition of a Lie group and of a Lie algebra.

Let G be a matrix Lie group. Show that the tangent space to G at the identity, equipped with an appropriate bracket, defines a Lie algebra $\mathcal{L}(G)$.

Let G be $SL(2,\mathbb{R})$, the group of real 2×2 matrices with unit determinant, where the group product is matrix multiplication. Explain briefly why G is a Lie group and find its Lie algebra $\mathcal{L}(G)$. Show that, for all $A\in\mathcal{L}(G)$, $A^2 = -\det(A)\mathbb{I}_2$ where $\mathbb{I}_2$ is the 2×2 unit matrix.

Define the exponential map Exp on $\mathcal{L}(G)$ for $G = SL(2,\mathbb{R})$ as an infinite series, and show that its image lies in G. Prove the lower bound,

$$\mathrm{Tr}\left[\mathrm{Exp}(A)\right] \;\; \geqslant \;\; -2$$

for all $A\in\mathcal{L}(G)$. Hence show that Exp is *not* surjective as a map $\mathcal{L}(G)\to G$.

2 Let $\mathfrak{g}$ be a simple, finite-dimensional, complex Lie algebra. State the definition of the *Cartan subalgebra* of $\mathfrak{g}$ and of the *roots* of $\mathfrak{g}$. Define a *Cartan-Weyl basis* of $\mathfrak{g}$ and give the general form of the brackets between basis elements. [*In the following you may assume that these definitions also hold for semi-simple Lie algebras.*]

Now let $\mathfrak{g} = \mathfrak{so}(4)$, the complexified Lie algebra of the Lie group $SO(4)$. The elements of $\mathfrak{g}$ are given as linear combinations of the 4×4 matrices $T^{(ij)} = -T^{(ji)}$ for $i > j = 1\ldots 4$ with entries,

$$\left(T^{(ij)}\right)_{\alpha\beta} \;\; = \;\; \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha} \qquad\qquad \alpha,\beta = 1,2,3,4$$

with brackets determined by the commutation relations,

$$[T^{(ij)},T^{(kl)}] \;\; = \;\; \delta_{jk}T^{(il)} + \delta_{il}T^{(jk)} - \delta_{jl}T^{(ik)} - \delta_{ik}T^{(jl)}$$

Choose a Cartan subalgebra for $\mathfrak{g}$ with basis elements $H^1 = T^{(12)}$ and $H^2 = T^{(34)}$. By considering the adjoint action of H^1 and H^2 on the linear combinations,

$$A^{\pm} \;\; = \;\; \left(T^{(13)} \pm T^{(24)}\right)$$
$$B^{\pm} \;\; = \;\; \left(T^{(14)} \pm T^{(23)}\right)$$

find the roots of $\mathfrak{g}$ and the corresponding step generators. Determine any non-vanishing brackets between the step generators.

What does it mean for two Lie algebras to be isomorphic? Construct an isomorphism between $\mathfrak{so}(4)$ and the direct sum $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ where $\mathfrak{su}(2)$ is the complexified Lie algebra of $SU(2)$. Identify the adjoint representation of $\mathfrak{so}(4)$ as a representation of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$.

3 *In this question you may use any general results for complex, simple, finite-dimensional Lie algebras and their representations as long as they are clearly and correctly stated.*

Let $\mathfrak{g}$ be a complex, simple, finite-dimensional Lie algebra of rank r with simple roots $\alpha^{(i)}$, $i = 1, 2, \ldots, r$. Define the *root lattice* and the *weight lattice* of $\mathfrak{g}$ and explain why the former is a sublattice of the latter. Define the Dynkin labels of a point in the weight lattice.

Draw Dynkin diagrams for all complex, simple, finite-dimensional Lie algebras of rank three. Label the nodes with an index $i = 1, 2, 3$ so that consecutive nodes are linked by at least one line and denote the corresponding simple roots as $\alpha^{(i)}$. The ordering should also be chosen so that simple roots $\alpha^{(1)}$ and $\alpha^{(2)}$ have the same length. In each case write down the corresponding Cartan matrix A and give the ratio $|\alpha^{(2)}|/|\alpha^{(3)}|$ as well as the angles between each pair of simple roots.

What is meant by the Dynkin labels of a finite-dimensional irreducible representation R of $\mathfrak{g}$? For each example of rank three, find the weights of the representation with Dynkin labels $[1, 0, 0]$. Here you should give the Dynkin labels of each weight of the representation. Comment on the dimensions of these representations.

4 Write an essay on *Non-Abelian gauge theory.* In your account you should construct in detail the Lagrangian for a gauge theory with gauge group G, coupled to matter transforming in a unitary representation of G, and discuss its invariance under infinitessimal gauge transformations. You should assume that G is a simple Lie group (ie one whose Lie algebra is simple) and explain the significance of this condition.

END OF PAPER