UNIVERSITY OF CAMBRIDGE

MAT3, MAMA, NST3AS, MAAS, NST3PHY, MAPY

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2019 9:00 am to 12:00 pm

PAPER 301

QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) The free Klein-Gordon field $\phi(\mathbf{x}, t)$ has Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 .$$

Using Noether's theorem, find an expression for the energy-momentum tensor of the theory. Thus find an expression for the total conserved 3-momentum $\mathbf{P}$ in terms of ϕ and its derivatives.

(ii) In the quantised Klein-Gordon theory and in the Schrödinger representation, the field $\phi(\mathbf{x})$ and the conjugate field $\pi(\mathbf{x})$ have the coupled expansions

$$\begin{aligned} \phi(\mathbf{x}) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \\ \pi(\mathbf{x}) &= \int \frac{d^3 p}{(2\pi)^3} (-i) \frac{\sqrt{E_p}}{\sqrt{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \end{aligned}$$

where $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ satisfy

$$\begin{aligned} \left[a_{\mathbf{p}},\ a_{\mathbf{p}'} \right] &= \left[a_{\mathbf{p}}^\dagger,\ a_{\mathbf{p}'}^\dagger \right] = 0 \\ \left[a_{\mathbf{p}},\ a_{\mathbf{p}'}^\dagger \right] &= (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'). \end{aligned}$$

The quantised operator $\mathbf{P}$ is implicitly normal-ordered.

(a) Determine $\mathbf{P}$ in terms of $a_{\mathbf{p}}$, $a_{\mathbf{p}}^\dagger$.

(b) Thus calculate $[\mathbf{P},\ a_{\mathbf{q}}^\dagger]$.

(c) Hence determine

$$e^{-i\mathbf{P}\cdot\mathbf{y}} a_{\mathbf{q}}^\dagger e^{i\mathbf{P}\cdot\mathbf{y}},$$

where $\mathbf{y}$ is a constant vector.

(d) What can you deduce about $e^{-i\mathbf{P}\cdot\mathbf{y}}|\mathbf{q}\rangle$, where $|\mathbf{q}\rangle$ denotes a one-particle state of 3-momentum $\mathbf{q}$?

(e) Find $e^{-i\mathbf{P}\cdot\mathbf{y}}\phi(\mathbf{x})e^{i\mathbf{P}\cdot\mathbf{y}}$, and interpret your result.

2 Consider the theory in 1+3 dimensions of a Dirac fermion ψ and a real scalar ϕ with Lagrangian density, where μ, m and λ are all real parameters and $\mu > 2m$:

$$\mathcal{L} = \bar{\psi}(i \not{\partial} - m)\psi + \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}\mu^2 \phi^2 - \lambda \bar{\psi}\gamma^\mu \psi \partial_\mu \phi.$$

(i) Draw and write momentum-space Feynman rules for the *interactions* of the theory.

(ii) What is the mass dimension of λ? What implication does this have for the theory?

(iii) Consider the decay $\phi(p) \to \bar{\psi}(q_1)\psi(q_2)$. Draw a tree-level Feynman diagram for the modulus squared of the amplitude. Using Feynman rules, calculate the tree-level width of ϕ, deriving any properties of γ matrices from the Clifford algebra which you should quote.

(iv) What is the spin averaged/summed cross-section for $\psi\bar{\psi} \to \psi\psi$ (write your reasoning)?

3

(i) Write down the defining relation of the Clifford algebra of γ^μ in 1+3 dimensions.

(ii) Write down the Dirac equation for a positive-frequency solution momentum space Dirac spinor $u(s, p^\mu)$ of spin s and 4-momentum p^μ.

(iii) Write down the Lagrangian density for an electron e^- of mass m and charge e coupled to an electromagnetic field A_μ.

(iv) Consider Compton scattering $\gamma(q, \epsilon_{in}(\lambda))e^-(p, s) \to \gamma(q', \epsilon_{out}(\lambda'))e^-(p', s')$ in Lorentz gauge.

(a) Write down/draw the Feynman rules needed for the calculation of the amplitude.

(b) Draw Feynman diagram(s) representing the total amplitude.

(c) Use the Feynman rules and diagrams to derive the total amplitude for Compton scattering.

(d) Calculate the modulus squared of the spin/polarisation averaged/summed amplitude in the limit $m \to 0$ in terms of the Mandelstam variables $s = (p+q)^2$ and $u = (p - q')^2$. Derive any properties of γ^μ you use from the Clifford algebra. You may assume that

$$\sum_s u(s,p)\bar{u}(s,p) = (\not{p} + m), \qquad \sum_\lambda \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) = -\eta_{\mu\nu}.$$

for a photon polarisation $\epsilon(\lambda)$.

4 This question regards a real scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_I(\phi).$$

(i) State Wick's theorem for $T\{\phi_1\phi_2\ldots\phi_n\}$, where $\phi_i = \phi(x_i)$, defining T and any other symbols you write.

(ii) Verify Wick's theorem for the scalar field for $n = 3$ assuming that it holds for $n = 2$.

(iii) For

$$\mathcal{L}_I(\phi) = -\frac{\lambda\phi^6}{6!},$$

calculate the vacuum to vacuum amplitude $\langle 0|T\{S\}|0\rangle$ up to $\mathcal{O}(\lambda^2)$ using Wick's theorem, where S is the $S-$matrix. Represent each term by a Feynman graph and show that, to $\mathcal{O}(\lambda^2)$, $\langle 0|T\{S\}|0\rangle$ may be represented diagrammatically by the exponential of the sum of distinct vacuum bubble types. Take care to write the combinatorial factors.

END OF PAPER