MAT3, MAMA

MATHEMATICAL TRIPOS   Part III

Thursday, 6 June, 2019 1:30 pm to 3:30 pm

PAPER 215

MIXING TIMES OF MARKOV CHAINS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a deck of n cards labeled 1 to n. The *random to top* card shuffling scheme is the Markov chain, whose state space is the $n!$ possible arrangements of the cards, that at each step picks a card uniformly at random (possibly the top card) and moves it to the top of the deck. The *top to random* card shuffling takes at each step the top card and insert it at a position chosen uniformly at random (possibly the top position).

Show that as a sequence in n both families of chains exhibit cutoff.

(You may use the solution to the coupon-collector problem as given, as well as the fact that shortly before the coupon collector time with a large probability many coupons are still not picked.)

2

(a) Let $d, n \in \mathbb{N}$. Let $e_1, \ldots, e_d$ be the standard basis of $(\mathbb{Z}/n\mathbb{Z})^d$. Let $p_1, \ldots, p_d \in [0, 1]$. Consider a random walk $X(k)$ on $(\mathbb{Z}/n\mathbb{Z})^d$, which evolves as follows: $X(k+1) - X(k)$ is distributed as $\sum_{i=1}^{d} e_i \xi_i$, where $\xi_1, \ldots, \xi_d$ are independent and ξ_i is equal to either $-1, 0$ or 1 with probability $\frac{1-p_i}{2}$, $1/2$ and $p_i/2$, respectively, for each i. Show that the mixing time is $O(n^2 \log(d+1))$, with the implicit constant being independent of the values of $p_1, \ldots, p_d$. Show for $d = 1$ a matching lower bound.

You may use without a proof the fact that for a ***simple*** random walk on the cycle, if ρ is the hitting time of 0 then $\max_{i \in \mathbb{Z}/n\mathbb{Z}} \mathbb{E}_i[\rho] \leqslant n^2$.

(b) Let $d, n \in \mathbb{N}$. Consider a random walk on $\{1, \ldots, n\}^d$, in which the d coordinates evolve independently, each according to the following transition matrix: for all $\ell \in \{2, \ldots, n-1\}$ let $P(\ell, \ell) = 1/2$ and $P(\ell, \ell+1) = 1/3 = 2P(\ell, \ell-1)$, whereas $P(0, 0) = 2/3 = 2P(0, 1)$ and $P(n, n) = 5/6 = 5P(n, n-1)$ (in other words, each co–ordinate attempts at each step to move up with probability $1/3$ and down with probability $1/6$, but moves away from the interval $\{1, \ldots, n\}$ are censored). We allow d to vary with n but require that $\lim_{n \to \infty} \frac{\log d}{n} = 0$. Show that the mixing time is $O(n)$.

(c) Show that the family of Markov chains from part (b) (indexed by n) exhibits cutoff. You may assume $d = 1$.

3

Let $G = (V, E)$ be a connected n-vertex simple graph (i.e. it has no self-loops nor multiple edges between a pair of vertices). Let $p \in (0, 1]$. Consider the following Markov chain on $\{\pm 1\}^V$: given that the state at time t is $\sigma_t \in \{\pm 1\}^V$ the next state σ_{t+1} is obtained by picking a vertex v uniformly at random, setting $\sigma_{t+1}(u) = \sigma_t(u)$ for all $u \neq v$ and then:

- With probability p set the value of $\sigma_{t+1}(v)$ to be 1 or -1 with equal probability.
- With probability $1-p$ a neighbor u of v (i.e. $uv \in E$) is picked uniformly at random and then we set $\sigma_{t+1}(v) = \sigma_t(u)$.

Let P be the corresponding transition matrix.

(i) Show that if G is regular (i.e. all vertices have the same degree), then for all $t \geqslant 0$ we have that

$$\max_{\sigma,\sigma' \in \{\pm 1\}^V} \|P^t(\sigma, \cdot) - P^t(\sigma', \cdot)\|_{\mathrm{TV}} \leqslant n \left(1 - \frac{p}{n}\right)^t .$$

(ii) Show that even if G is not regular, for all $t \geqslant 0$ we have that

$$\max_{\sigma,\sigma' \in \{\pm 1\}^V} \|P^t(\sigma, \cdot) - P^t(\sigma', \cdot)\|_{\mathrm{TV}} \leqslant 2|E| \left(1 - \frac{p}{n}\right)^t .$$

[*Hint: the forms of the bounds suggest what method should be used.*]

4

(a) State and prove a theorem demonstrating the use of the canonical paths method to obtain a Poincaré inequality.

(b) Consider a Markov chain $(X_t)_{t \geqslant 0}$ with a stationary distribution π. Let A be a subset of the state space Ω with $\pi(A) \geqslant 1/2$. Denote its hitting time by $T_A := \inf\{t : X_t \in A\}$. Show that the 1/4 total-variation mixing time is at least $c \max_{x \in \Omega} \mathbb{E}_x[T_A]$, where c is some positive absolute constant independent of the Markov chain.

END OF PAPER