MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2019 9:00 am to 11:00 am

PAPER 214

PERCOLATION AND RANDOM WALKS ON GRAPHS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $(x_n)_{n \geqslant 0}$ be a real sequence satisfying $x_{n+m} \leqslant x_n + x_m$ for all $n, m \geqslant 0$. Prove that the limit of x_n/n as $n \to \infty$ exists and satisfies

$$\lim_{n\to\infty} \frac{x_n}{n} = \inf \left\{ \frac{x_n}{n} \right\}.$$

(b) Let $d \geqslant 2$ and consider bond percolation on $\mathbb{Z}^d$ with $p \in (0,1]$. For each m denote $B(m) = [-m,m]^d \cap \mathbb{Z}^d$ and $\partial B(m) = \{(x_1,\ldots,x_d) \in B(m) : \exists\, i \text{ s.t. } x_i \in \{-m,m\}\}$.

(i) For each $n \in \mathbb{N}$ let $e_n = (n,0,\ldots,0)$. Prove that for each p there exists $\phi(p)$ such that

$$\lim_{n\to\infty} \left(-\frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow e_n) \right) = \phi(p).$$

(ii) Prove that for each $n \in \mathbb{N}$ there exists $x \in \partial B(n)$ with $x_1 = n$ such that

$$\mathbb{P}_p(0 \leftrightarrow x) \geqslant \frac{1}{|\partial B(n)|} \mathbb{P}_p(0 \leftrightarrow \partial B(n)).$$

(iii) By relating $(\mathbb{P}_p(0 \leftrightarrow \partial B(m)))_m$ to $(\mathbb{P}_p(0 \leftrightarrow e_n))_n$ or otherwise prove that

$$\lim_{n\to\infty} \left(-\frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow \partial B(n)) \right) = \phi(p).$$

(You may use without proof that there exist positive constants c_1 and c_2 so that $c_1 n^{d-1} \leqslant |\partial B(n)| \leqslant c_2 n^{d-1}$ for all n.)

2

Let $d \geqslant 2$ and consider bond percolation on $\mathbb{Z}^d$ with $p \in (0,1]$. For each m denote $B(m) = [-m,m]^d \cap \mathbb{Z}^d$ and $\partial B(m) = \{(x_1,\ldots,x_d) \in B(m) : \exists\, i \text{ s.t. } x_i \in \{-m,m\}\}$. Denote by p_c the critical probability.

(a) If $p < p_c$, show that there exists a positive constant c so that for all $n \geqslant 1$ we have

$$\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_n) \leqslant e^{-cn}.$$

(You may use without proof that

$$\widetilde{p_c} = \sup\{p \in [0,1] : \exists \text{ a finite set } S \ni 0 \text{ with } \phi_p(S) < 1\},$$

where $\phi_p(S) = p \sum_{(x,y)\in\partial S} \mathbb{P}_p\left(0 \stackrel{S}{\longleftrightarrow} x\right)$, satisfies $\widetilde{p_c} = p_c$.)

(b) Let $\mathcal{C}$ be the open cluster of 0, i.e. the set of vertices of $\mathbb{Z}^d$ connected to 0 via open paths of edges. Let $\chi(p) = \mathbb{E}_p[|\mathcal{C}|]$. Show that

$$\chi(p) < \infty \quad \text{if and only if} \quad p < p_c.$$

3

Let $G = (V,E)$ be a finite connected graph with conductances $(c(e))_e$ on the edges. Let a and b be two distinguished vertices. Let X be a weighted random walk on G with edge weights $(c(e))_e$. For every $x \in V$ define $\tau_x = \min\{t \geqslant 0 : X_t = x\}$.

(a) Define the terms: voltage, current flow and effective resistance between a and b.

(b) Let $x \in V \setminus \{a,b\}$. Show that

$$\mathbb{P}(\tau_a < \tau_b \mid X_0 = x) \leqslant \frac{R_{\text{eff}}(x,\{a,b\})}{R_{\text{eff}}(x,a)}.$$

(Hint: Consider excursions from x back to x and take the first one that hits $\{a,b\}$.)

(c) Prove that the effective resistance satisfies

$$\frac{1}{R_{\text{eff}}(a,b)} = \inf\left\{\frac{1}{2}\sum_{x,y}(f(x)-f(y))^2 c(x,y) : \, f(a)=1 \text{ and } f(b)=0\right\},$$

where the infimum is over all functions satisfying the two boundary conditions.

(Hint: The three parts are unrelated.)

4

(a) Define the term uniform spanning tree of a finite connected graph G.

(b) Describe Wilson's algorithm for generating a uniform spanning tree of a finite connected graph G.

Explain the meaning of the term "uniform spanning tree of an infinite recurrent graph" and describe a way of generating it.

(c) Let T be a uniform spanning tree of $\mathbb{Z}^2$. Show that if e is an edge of $\mathbb{Z}^2$, then

$$\mathbb{P}(e \in T) = \frac{1}{2}.$$

(Hint: Show that the expected degree of a vertex in T is 2 by considering an exhaustion of $\mathbb{Z}^2$ by balls.)

END OF PAPER