MAT3, MAMA, MGM3

# MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 9:00 am to 12:00 pm

## PAPER 211

## ADVANCED FINANCIAL MODELS

*Attempt no more than* ***FOUR*** *questions.*

*There are* ***SIX*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***

*Cover sheet*

*Treasury Tag*

*Script paper*

*Rough paper*

***SPECIAL REQUIREMENTS***

*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $X$ be a given an $n$-dimensional random vector. Throughout this question, you may *not* use a fundamental theorem of asset pricing without proof.

(a) Suppose that $H \in \mathbb{R}^n$ is such that $H \cdot X \geqslant 0$ almost surely and $\mathbb{P}(H \cdot X > 0) > 0$. Prove that there does not exists a positive random variable $\rho$ such that $\mathbb{E}(\rho) = 1$ and $\mathbb{E}(\rho X) = 0$.

Given a positive random variable $\zeta$, define a function $F$ on $\mathbb{R}^n$ by

$$F(h) = \mathbb{E}[e^{-h \cdot X} \zeta].$$

Suppose $F$ is everywhere finite and smooth. Let

$$f = \inf_{h \in \mathbb{R}^n} F(h).$$

A sequence $(h_k)_k$ such that $F(h_k) \to f$ is called a minimising sequence.

(b) Suppose there exists a bounded minimising sequence. Show that there exists a positive random variable $\rho$ such that $\mathbb{E}(\rho) = 1$ and $\mathbb{E}(\rho X) = 0$.

(c) Suppose every minimising sequence is unbounded. Show that there exists a vector $H \in \mathbb{R}^n$ such that $H \cdot X \geqslant 0$ almost surely and $\mathbb{P}(H \cdot X > 0) > 0$.

(d) Suppose that there exists a unique positive random variable $\rho$ such that $\mathbb{E}(\rho) = 1$ and $\mathbb{E}(\rho X) = 0$. Show that every random variable $Y$ is of the form $Y = a + b \cdot X$ for constants $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$. [Hint: Use part (b) twice with $\zeta = e^{cY - Y^2 - \|X\|^2}$ for $c = 0$ and $c = 1$.]

**2** Let $S$ be a positive random variable such that $\mathbb{E}(S) = 1$. Let $X$, $Y$ and $Z$ have the unit exponential $f_X(x) = e^{-x}1_{\{x>0\}}$, Cauchy $f_Y(y) = \frac{1}{\pi}(y^2+1)^{-1}$ and standard normal $f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ densities, respectively.

(a) Prove that

$$M(\theta) = \mathbb{E}(e^{\theta \log S})$$

is well-defined and bounded for all $\theta \in \{p + \mathrm{i}q : 0 \leqslant p \leqslant 1, q \in \mathbb{R}\}$, where $\mathrm{i} = \sqrt{-1}$.

(b) Compute $M(\theta)$ in the case where $S = (1+t)e^{-tX}$, for a constant $t > 0$.

(c) Prove the identity

$$\mathbb{E}[(S-K)^+] = 1 - \sqrt{K}\mathbb{E}\big[M\big(\tfrac{1}{2}(1+\mathrm{i}Y)\big)e^{-\frac{1}{2}\mathrm{i}Y\log K}\big] \text{ for all } K > 0.$$

Explain briefly why the above identity is useful in the context of a stochastic volatility model such as the Heston model. You may use without proof the identity

$$\mathbb{E}(e^{\mathrm{i}Yt}) = e^{-|t|} \text{ for all } t \in \mathbb{R}.$$

(c) Let $G(S) = e^{-\frac{1}{2}(\log S)^2}$. Prove that

$$\mathbb{E}[G(S)] = \mathbb{E}[M(\mathrm{i}Z)]$$

where $Z \sim N(0,1)$. You may use without proof the identity

$$\mathbb{E}(e^{\mathrm{i}tZ}) = e^{-t^2/2}.$$

**3**

Consider a continuous time model of a market with two assets with positive prices $(B, S)$ where

$$\begin{aligned} dB_t &= B_t \; r_t \; dt, \\ dS_t &= S_t(\mu_t \; dt + \sigma_t \; dW_t), \end{aligned}$$

where each of the processes $r$, $\mu$ and $\sigma$ are adapted, positive and continuous, and where $W$ is a Brownian motion which generates the filtration $\mathcal{F}$.

(a) Let $Y$ be a local martingale deflator with $Y_0 = 1$. Show that

$$dY_t = -Y_t(r_t \; dt + \lambda_t \; dW_t)$$

for an adapted, continuous process $\lambda$ to be determined in terms of the given processes $r$, $\mu$ and $\sigma$.

(b) Fix a non-random $T$, and let $\xi_T$ be a non-negative, bounded $\mathcal{F}_T$-measurable random variable. Show that there exists an admissible pure-investment trading strategy $(\phi_t, \pi_t)_{0 \leqslant t \leqslant T}$ such that $\phi_T B_T + \pi_T S_T = \xi_T$ almost surely. Show that the minimal initial cost among all such replication strategies is

$$\phi_0 B_0 + \pi_0 S_0 = \mathbb{E}(Y_T \xi_T)$$

where $Y$ is the local martingale deflator from part (a). [You may use standard results from stochastic calculus if clearly stated.]

Now suppose that $r$, $\mu$ and $\sigma$ are positive constants, and suppose that the payout of the claim in part (b) is of the form $\xi_T = g(S_T)$. Let $(\phi, \pi)$ be the minimal cost replicating portfolio.

(c) Show that there exists a function $V : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ with the property that

$$V(t, S_t) = \phi_t B_t + \pi_t S_t \text{ for all } 0 \leqslant t \leqslant T.$$

(d) Assuming that the function $V$ in part (c) is smooth, show that there is a function $\tilde{V}$ with the property that $\pi_t = \tilde{V}(t, S_t)$. How is $\tilde{V}$ related to $V$?

**4** Consider a discrete-time market with $n$ assets with (possibly negative) prices $(P_t)_{t\geqslant 0}$.

(a) What is an investment-consumption arbitrage? What is a terminal-consumption arbitrage?

(b) What is a numéraire strategy? Prove that if the market has an investment-consumption arbitrage and a numéraire strategy, then the market has a terminal consumption arbitrage.

(c) Suppose there exists a non-negative adapted process $(Z_t)_{t\geqslant 0}$ such that

$$\mathbb{P}(Z_t = 0 \text{ for all } t) = 0$$

and such that the process $M$ defined by

$$M_t = (-1)^t Z_t P_t$$

is a martingale. Prove that the market has no numéraire strategies.

[You may use without proof the standard properties of local martingales discussed in lectures. You may also also use the fact that if $(X_t)_{t\geqslant 0}$ is a local martingale with respect to a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ then $(X_{t_k})_{k\geqslant 0}$ is a local martingale with respect to the filtration $(\mathcal{F}_{t_k})_{k\geqslant 0}$ for any increasing (non-random) sequence $(t_k)_{k\geqslant 0}$.]

**5** Consider a discrete-time market model with prices $(P_t^T)_{t\in[0,T],T\geqslant 1}$ where $P_t^T$ is the price at time $t$ of a risk-free zero-coupon bond of unit face value and maturity $T$. Assume that the prices are adapted to a filtration $(\mathcal{F}_t)_{t\geqslant 0}$, and that the market is free of arbitrage.

(a) Explain why $P_t^T > 0$ almost surely for all $0 \leqslant t \leqslant T$.

(b) Define the spot interest rate $r_t$ in terms of the bond prices. Define the bank account $B_t$ in terms of the spot interest rate. What does it mean to say a probability measure $\mathbb{Q}$ is a risk-neutral measure for this model?

(c) Show that $T \mapsto P_t^T$ is non-increasing almost surely for all $t$ if and only if $r_t \geqslant 0$ almost surely for all $t \geqslant 0$.

(d) Suppose the spot interest rate $(r_t)_{t\geqslant 1}$ evolves as

$$1 + r_t = \zeta_{t-1}(1 + r_{t-1})$$

where where $(\zeta_t)_{t\geqslant 1}$ is a sequence of positive independent and identically distributed random variables generating the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. For exponents $n \in \mathbb{Z}$, let

$$M(n) = \mathbb{E}^{\mathbb{Q}}(\zeta_1^n)$$

for a fixed risk-neutral measure $\mathbb{Q}$, and assume $M$ is finite-valued. Compute the bond price $P_t^T$ in terms of the function $M$.

**6** Consider a market with two assets, a bank account with time-$t$ price $e^{rt}$ and a stock whose price dynamics satisfy

$$\begin{aligned} dS_t &= S_t(r\ dt + \sqrt{v_t}dW_t) \\ dv_t &= (a - bv_t)dt + c\sqrt{v_t}(\rho dW_t + \sqrt{1-\rho^2}dZ_t) \end{aligned}$$

where $r, a, b, c$ and $\rho$ are contants, with $a, b > 0$ and $-1 \leqslant \rho \leqslant 1$, and $W$ and $Z$ are independent Brownian motions.

Let $F : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the partial differential equation

$$\frac{\partial F}{\partial t} + Sr\frac{\partial F}{\partial S} + (a - bv)\frac{\partial F}{\partial v} + \frac{1}{2}S^2 v\frac{\partial^2 F}{\partial S^2} + c\rho S v\frac{\partial^2 F}{\partial S \partial v} + \frac{1}{2}c^2 v\frac{\partial^2 F}{\partial v^2} = rF$$

with boundary condition $F(T, S, v) = \sqrt{S}$.

Introduce a contingent claim with time-$T$ payout $\xi_T = \sqrt{S_T}$.

(a) Show that there is no arbitrage relative to the bank account in the augmented market consisting of the bank account, stock and contingent claim, if the time-$t$ price of the contingent claim is given by $\xi_t = F(t, S_t, v_t)$. You may use a fundamental theorem of asset pricing as long as it is stated carefully.

Suppose that $F(t, S, v) = \sqrt{S}e^{A(t)v+B(t)}$ for some functions $A, B : [0, T] \to \mathbb{R}$.

(b) Show that $A$ satisfies an ordinary differential equation. You should derive the equation, including the boundary conditions, but need not solve it.

(c) Show that the function $B$ is given by

$$B(t) = -\tfrac{1}{2}(T - t)r + k\int_t^T A(s)ds$$

for a constant $k$ which you should find in terms of the model parameters.

**END OF PAPER**