UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2019 9:00 am to 11:00 am

PAPER 203

SCHRAMM-LOEWNER EVOLUTIONS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of a *compact* $\mathbb{H}$*-hull* A and its *half-plane capacity* $\mathrm{hcap}(A)$.

(b) (i) Prove that $\mathrm{hcap}(A) = \lim_{y\to\infty} y\mathbb{E}_{iy}\mathrm{Im}[B_\tau]$ where $\tau = \inf\{t \geqslant 0 : B_t \notin \mathbb{H} \setminus A\}$, B is a complex Brownian motion, and $\mathbb{E}_{iy}$ denotes the expectation under the law where $B_0 = iy$. *[You may assume without proof that* $\mathrm{hcap}(A)$ *is real.]*

(ii) Prove that hcap is monotone: for compact $\mathbb{H}$-hulls A, C, if $A \subseteq C$, then $\mathrm{hcap}(A) \leqslant \mathrm{hcap}(C)$.

(iii) Prove or disprove: for compact $\mathbb{H}$-hulls A, C, if $A \subseteq C$ and $\mathrm{hcap}(A) = \mathrm{hcap}(C)$ then $A = C$.

(c) Explain what it means for a family (A_t) of compact $\mathbb{H}$-hulls indexed by $t \geqslant 0$ to be:

(i) *non-decreasing*;

(ii) *parameterized by capacity*;

(iii) satisfy the *local growth property.*

(d) Consider the compact $\mathbb{H}$-hulls (A_t) given by $A_t = \sqrt{2t}(\mathbb{H} \cap \overline{\mathbb{D}})$. Determine (with proof) which of properties (i)–(iii) from part (c) hold for (A_t). *[You may use without proof that the unique conformal transformation* $\psi\colon \mathbb{H} \setminus \overline{\mathbb{D}} \to \mathbb{H}$ *with* $\psi(z) - z \to 0$ *as* $z \to \infty$ *is given by* $\psi(z) = z + 1/z$*.]*

2

(a) Suppose that γ is an SLE_κ in $\mathbb{H}$ from 0 to ∞ and $r > 0$. Prove that $t \mapsto r\gamma(t/r^2)$ has the same distribution as γ.

(b) Explain how SLE_κ connecting two distinct boundary points in a simply connected domain is defined. Prove also that the definition is well-defined.

(c) Fix $\kappa \in (0, 8)$. Let (g_t) be the solution to the Loewner equation driven by $U_t = \sqrt{\kappa}B_t$ where B is a standard Brownian motion. Let

$$M_t(z) = \Upsilon_t^{(\kappa-8)/8} S_t^{(8-\kappa)/\kappa}$$

where $z_t = x_t + iy_t = g_t(z)$, $\Upsilon_t = y_t/|g_t'(z)|$, and $S_t = \sin(\arg(z_t - U_t))$. For each $\epsilon > 0$, let $\tau_\epsilon = \inf\{t \geqslant 0 : \Upsilon_t = \epsilon\}$. *[You may use without proof that M_t is a continuous local martingale.]*

(i) Prove that $M_{t\wedge\tau_\epsilon}$ is a bounded martingale. *[You may use results from lectures provided you state them clearly.]*

(ii) Suppose that $K \subseteq \mathbb{H}$ is compact. Show that there exists $\epsilon_0 > 0$ so that $\Upsilon_0 \geqslant \epsilon_0$ for all $z \in K$. Show that there exists a constant $c_0 > 0$ so that $\mathbb{P}[\tau_\epsilon < \infty] \leqslant c_0 \epsilon^{(8-\kappa)/\kappa}$ for all $z \in K$ and $\epsilon \in (0, \epsilon_0)$. *[You may use without proof that there exists a constant $c_1 > 0$ so that $\mathbb{P}[S_{\tau_\epsilon} \geqslant 1/2 \,|\, \tau_\epsilon < \infty] \geqslant c_1$ for all $z \in K$.]*

(iii) Show that the range of an SLE_κ with $\kappa \in (0, 8)$ in $\mathbb{H}$ from 0 to ∞ a.s. has zero Lebesgue measure. *[Hint: bound the expected Lebesgue measure of the set of points $z \in K$ for which $\tau_\epsilon < \infty$.]*

3

(a) Specify for which range of κ values SLE_κ is simple, is self-intersecting but not space-filling, and is space-filling.

(b) Prove that SLE_κ is simple for the range of κ values that you have specified. *[You may use properties of Bessel processes proved in class provided you state them clearly.]*

(c) Suppose that γ is an SLE_4 in $\mathbb{H}$ from 0 to ∞, let (g_t) be its asociated Loewner flow, and U_t its Loewner driving function. Fix $z \in \mathbb{H}$.

(i) Prove that $\log(g_t(z) - U_t)$ is a continuous local martingale.

(ii) Deduce that the probability that γ passes to the right of z is given by $\frac{1}{\pi}\arg(z)$.

[You may assume that z is a.s. not in the range of γ.]

(d) Suppose that γ is a simple curve in $\mathbb{H}$ from 0 to ∞ which is parameterized by capacity. Let U_t be its Loewner driving function, (g_t) be the associated family of conformal maps, and $\mathcal{F}_t = \sigma(\gamma(s) : s \leqslant t) = \sigma(U_s : s \leqslant t)$. Assume that $\log(g_t(z) - U_t)$ is a $(\mathcal{F}_t)$-local martingale for every $z \in \mathbb{H}$.

(i) Show that U_t is a semimartingale.

(ii) Deduce that γ is an SLE_4. *[Hint: apply Itô's formula to $\log(g_t(z) - U_t)$, use that the drift term must vanish for every $z \in \mathbb{H}$, and examine the behavior of the drift term for $z \in \mathbb{H} \setminus \gamma([0,t])$ close to $\gamma(t)$.]*

4 Let $\mathcal{Q}_\pm$ be the set of compact $\mathbb{H}$-hulls A with $0 \notin \overline{A}$ and for each $A \in \mathcal{Q}_\pm$ let ψ_A be the unique conformal transformation $\mathbb{H} \setminus A \to \mathbb{H}$ with $\psi_A(0) = 0$ and $\lim_{z\to\infty} \psi_A(z)/z = 1$. Let γ be an $\mathrm{SLE}_{8/3}$ in $\mathbb{H}$ from 0 to ∞.

(a) State what it means for the law of γ to satisfy the *restriction property.*

(b) Prove that the law of γ satisfies the restriction property if the following is true. There exists $\alpha > 0$ so that for every $A \in \mathcal{Q}_\pm$ we have that

$$\mathbb{P}[\gamma \cap A = \emptyset] = (\psi_A'(0))^\alpha.$$

(c) Let U_t be the Loewner driving function for γ, (g_t) the associated family of conformal maps, $A \in \mathcal{Q}_\pm$, $\widetilde{g}_t = g_{\psi_A(\gamma[0,t])}$, and $\psi_t = \widetilde{g}_t \circ \psi_A \circ g_t^{-1}$. Let $M_t = (\psi_t'(U_t))^\alpha$ and $\tau = \inf\{t \geqslant 0 : \gamma(t) \in A\}$. Prove for a choice of α you should identify that $M_{t\wedge\tau}$ is a continuous local martingale. *[You may use without proof the formula*

$$(\partial_t \psi_t)'(U_t) := \lim_{z\to U_t} \partial_t \psi_t'(z) = \frac{(\psi_t''(U_t))^2}{2\psi_t'(U_t)} - \frac{4}{3}\psi_t'''(U_t) \quad .]$$

Explain further why $M_{t\wedge\tau}$ is in fact a bounded martingale. *[You may use results from lectures provided you state them clearly.]*

(d) Compute the probability of:

(i) $\gamma[0,\infty) \cap (\mathbb{H} \cap \overline{B(1,\epsilon)}) = \emptyset$ for $\epsilon \in (0,1)$.

(ii) $\gamma[0,\infty) \cap (1, 1 + i\epsilon] = \emptyset$ for each $\epsilon > 0$.

END OF PAPER