MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2019 9:00 am to 12:00 am

PAPER 202

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

1. Let M be a continuous local martingale that is bounded by an integrable random variable Z, i.e., $\sup_t |M_t| \leqslant Z$. Show that M is a martingale. Is M uniformly integrable?

2. Let B be a standard Brownian motion, and let H be a continuous adapted process with $\int_0^\infty H_s^2\, ds = \infty$ almost surely. For $\sigma > 0$, let $T_\sigma = \inf\{t : \int_0^t H_s^2\, ds > \sigma^2\}$. Find the distribution of the random variable

$$X_\sigma = \int_0^{T_\sigma} H_s\, dB_s.$$

[If you use the Dubins–Schwarz theorem, you must prove it.]

3. Let B be a standard Brownian motion with $B_0 = 0$. For $a > 0$, prove that

$$\mathbb{P}(\sup_{t \leqslant s} |B_t| \geqslant a) \leqslant 2e^{-\frac{a^2}{2s}}.$$

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function such that $|f(x)| + |\nabla f(x)|^2 + |\Delta f(x)| \leqslant C|x|^{2-\epsilon}$ for some $C, \epsilon > 0$. Prove that

$$M_t = \exp\left(f(B_t) - \frac{1}{2}\int_0^t (|\nabla f(B_s)|^2 + \Delta f(B_s))\, ds \right)$$

is a (true) martingale.

1. Let M be an bounded martingale with $|M_t| \leqslant C$ for all t where C is a deterministic constant. Let $0 = t_0 < t_1 < \cdots < t_n = t$. Prove that

$$\mathbb{E}\left(\left(\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2\right)^2\right) \leqslant 48C^4.$$

2. Let $b : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\|b'\|_\infty < \infty$. Prove that for every continuous function $w : \mathbb{R}_+ \to \mathbb{R}$ and every $x_0 \in \mathbb{R}$, there is a unique continuous solution $x : \mathbb{R}_+ \to \mathbb{R}$ to the integral equation

$$x(t) = x_0 + \int_0^t b(x(s))\, ds + w(t).$$

Now let $(W_t)_{t\geqslant 0}$ is a continuous stochastic process and let $\mathcal{F}_t = \sigma(W_s : s \leqslant t)$. Define $X = (X_t)_{t\geqslant 0}$ by $X_t = x(t)$ where $x(t)$ is the solution above with $w = W$. Show that X is adapted with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

3. Suppose that $H : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies, for some constant C,

$$H(x) \to \infty \quad (|x| \to \infty), \qquad -\frac{1}{2}H''(x) + H'(x)^2 \geqslant -C \quad (x \in \mathbb{R}).$$

Let X be the local solution to the SDE

$$dX_t = -H'(X_t)\, dt + dB_t, \quad X_0 = x.$$

Prove that this SDE has a global solution, i.e., the explosion time T is $+\infty$ a.s. [You may use the characterisation of the explosion time proved in the lectures.]

[*Hint: Consider* $H(X_t)$.]

3

1. Define the terms strong solution, weak solution, uniqueness in law, and pathwise uniqueness for an SDE.

2. Let $a \geqslant 0$. Set $T_a = \inf\{s \geqslant a : B_s = 0\}$. Show that

$$X_t = \begin{cases} 0 & (t < T_a) \\ B_t^3 & (t \geqslant T_a) \end{cases}$$

is a strong solution to the SDE

$$dX_t = 3\mathrm{sign}(X_t)|X_t|^{1/3} + 3|X_t|^{2/3}\, dB_t.$$

Conclude that pathwise uniqueness does not hold for this SDE.

3. State and prove the Feynman–Kac formula for $X = B$ a standard Brownian motion.

4. Use the Feynman–Kac formula to compute

$$\mathbb{E}_x(e^{-\sigma \int_0^t B_s\, ds})$$

where $B_0 = x$ under the law $\mathbb{P}_x$ for which $\mathbb{E}_x$ is the expectation.

[Hint: the following ansatz might be useful: $e^{A(t)x+B(t)}$. You are allowed to assume uniqueness of any differential equation that might be useful.]

4

1. Let M and N be continuous local martingales, and let $t > s$. Prove that

$$|\langle M, N\rangle_t - \langle M, N\rangle_s| \leqslant \sqrt{\langle M, M\rangle_t - \langle M, M\rangle_s}\sqrt{\langle N, N\rangle_t - \langle N, N\rangle_s}.$$

2. Find a weak solution to the SDE

$$dX_t = \mathrm{sign}(X_t)\, dB_t, \quad X_0 = 0, \quad \mathrm{sign}(x) = \begin{cases} +1 & (x > 0) \\ -1 & (x \leqslant 0). \end{cases}$$

Show that the SDE does not have a strong solution. You may use the following result without proof: For every continuous semimartingale X, there exists a continuous increasing process (L_t) such that

$$|X_t| = |X_0| + \int_0^t \mathrm{sign}(X_s)\, dX_s + L_t$$

and L is adapted to the completed filtration of $|X|$.

3. Let $\frac{1}{2}\sigma^2 \neq \beta$, and let X be unique strong solution to the SDE

$$dX_t = \beta X_t\, dt + \sigma X_t\, dB_t, \quad X_0 = x > 0. \qquad (*)$$

Let $T_s = \inf\{t \geqslant 0 : X_t = s\}$ and assume that $x \in (r, R)$, where $r > 0$. Compute $\mathbb{P}(T_r < T_R)$.

[Hint: Find the generator L associated to $(*)$ and compute Lf for $f(x) = x^\gamma$ where $\gamma \in \mathbb{R}$ is a constant.]

5

1. Define what it means for a process to be a simple process. Let H be a simple process, and let B be a standard Brownian motion with $B_0 = 0$. Define $H \cdot B$, show that $H \cdot B$ is a martingale, and prove that $(H \cdot B)_t^2 - \int_0^t H_s^2 \, ds$ is a martingale.

2. Let $f : [0, \infty) \to \mathbb{R}$ be a deterministic continuous function. Prove that

$$\mathbb{E}\left(B_t \int_0^t f(s) \, dB_s \right) = \int_0^t f(s) \, ds.$$

3. Let $\mu, \sigma : [0, \infty) \to \mathbb{R}$ be deterministic continuous functions, assume that σ is bounded below by a strictly positive constant, and that μ has compact support. Assume that X is a solution to

$$dX_t = X_t(\mu(t) \, dt + \sigma(t) \, dB_t), \quad X_0 = 1,$$

with respect to the probability measure $\mathbb{P}$.

Prove that $X_t e^{-\int_0^t \mu(s) \, ds}$ is a local martingale under $\mathbb{P}$.

Find a probability measure $\mathbb{Q}$ such that X is a local martingale.

6

1. Show that a continuous martingale that is almost surely of finite variation is constant.

2. Let B and $\tilde{B}$ be independent standard Brownian motions defined on the same probability space with $B_0 = \tilde{B}_0 = 0$. Let

$$X_t = e^{B_t} \int_0^t e^{-B_s} \, d\tilde{B}_s, \quad Y_t = \sinh B_t.$$

Show that X and Y have the same law. [You may use that SDEs with Lipschitz coefficients satisfy the uniqueness in law property.]

3. Solve the SDE

$$dX_t = (-aX_t + b) \, dt + \sigma \, dB_t, \quad X_0 = x,$$

and compute $\mathrm{cov}(X_t, X_s)$ for all $t, s > 0$.

END OF PAPER