MAT3, MAMA, EGT6

**MATHEMATICAL TRIPOS Part III**

Monday, 3 June, 2019 1:30 pm to 4:30 pm

# PAPER 201

# ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Fix $1/2 < p < 1$. Let $X_1, X_2, \cdots$ be independent identically distributed random variables with

$$\mathbb{P}(X_1 = 1) = p \quad \text{and} \quad \mathbb{P}(X_1 = -1) = 1 - p.$$

Let $S_n = X_1 + \cdots + X_n$ and $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$.

(a) Find $\lambda \in (0, 1)$ so that $\lambda^{S_n}$ is martingale for $(\mathcal{F}_n)$.

In the following, we let $\varphi(x) = \lambda^x$ for the value of $\lambda$ found in (a). By convention, $\varphi(0) = 1$.

(b) Let $T_x = \inf\{n : S_n = x\}$. Use the martingale $\varphi(S_n)$ to show that for $a < 0 < b$, where $a, b \in \mathbb{Z}$, we have

$$\mathbb{P}(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}.$$

(c) Deduce that for $a < 0 < b$, where $a, b \in \mathbb{Z}$, we have

$$\mathbb{P}(T_a < \infty) = \lambda^{-a} \quad \text{and} \quad \mathbb{P}(T_b < \infty) = 1.$$

(d) Is $\varphi(S_n)$ uniformly integrable? Justify your answer.

**2** Let $S_n = X_1 + \cdots + X_n$ where $X_1, X_2, \cdots$ are independent and $\mathbb{E}(X_m) = 0$, $\mathbb{E}(X_m^2) = \sigma_m^2 \in (0, \infty)$ for all $m \geqslant 1$. Let $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$.

(a) Show that for $c > 0$, $(S_n + c)^2$ is a submartingale for $(\mathcal{F}_n)$.

(b) Use the submartingale in (a) and Doob's inequality to show that for all $x > 0$

$$\mathbb{P}\left(\max_{1 \leqslant m \leqslant n} S_m \geqslant x\right) \leqslant \frac{\operatorname{var}(S_n)}{\operatorname{var}(S_n) + x^2}.$$

(c) Show that $S_n^2 - \operatorname{var}(S_n)$ is a martingale for $(\mathcal{F}_n)$.

(d) Now, suppose in addition that $|X_m| \leqslant K$ a.s. for all $m \geqslant 0$. Use the martingale in (c) to show that for all $x > 0$

$$\mathbb{P}\left(\max_{1 \leqslant m \leqslant n} |S_m| \leqslant x\right) \leqslant \frac{(x + K)^2}{\operatorname{var}(S_n)}.$$

**3** Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, each uniformly distributed on the set $\{-1, 1\}$. Set $S_n = X_1 + \cdots + X_n$.

(a) Compute the cumulant generating function $\psi$ of $X_1$ and show that its Legendre transform is given by

$$\psi^*(x) = \frac{(1+x)\log(1+x) + (1-x)\log(1-x)}{2}.$$

(b) Consider for $0 \leqslant a < c \leqslant 1$ the event $A_n = \{an \leqslant S_n \leqslant cn\}$. Show that, for all $\lambda \geqslant 0$, we have

$$\mathbb{P}(A_n) \geqslant e^{-\lambda cn + \psi(\lambda)n} \mathbb{P}_\lambda(A_n)$$

where $\mathbb{P}_\lambda$ is the tilted probability measure, given by

$$d\mathbb{P}_\lambda / d\mathbb{P} = e^{\lambda S_n - \psi(\lambda)n}.$$

(c) Deduce by suitable choices of $c$ and $\lambda$ that, for all $a \in [0, 1)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geqslant an) \geqslant -\psi^*(a).$$

**4**

(a) What does it mean to say that a random process $(X_t)_{t\geqslant 0}$ is a Brownian motion in $\mathbb{R}^d$?

(b) Show that, if $(X_t)_{t\geqslant 0}$ is a Brownian motion in $\mathbb{R}^d$ and if $U$ is an orthogonal $d \times d$ matrix, then the process $(UX_t)_{t\geqslant 0}$ is also a Brownian motion in $\mathbb{R}^d$.

(c) Let $D$ be a domain in $\mathbb{R}^d$ and let $A$ be a measurable subset of its boundary $\partial D$. For $x \in D$, define

$$\phi(x) = \mathbb{P}(X_T \in A)$$

where $(X_t)_{t\geqslant 0}$ is a Brownian motion in $\mathbb{R}^d$ starting from $x$ and

$$T = \inf\{t \geqslant 0 : X_t \notin D\}.$$

Show that $\phi$ is a harmonic function on $D$.

(d) Find the function $\phi$ in the case $d = 2$ for

$$D = \{(x, y) : x \in \mathbb{R}, y > 0\}, \quad A = \{(x, 0) : x > 0\}.$$

**5** Let $(X_t)_{t\geqslant 0}$ be a Brownian motion in $\mathbb{R}^2$. Let $f$ be a continuous bounded probability density function on $\mathbb{R}^2$ and set

$$A_t = \int_0^t f(X_s)ds.$$

(a) Show that the set of times $\{t \geqslant 0 : |X_t| \leqslant 1\}$ is unbounded almost surely.

(b) Show that $\mathbb{E}(A_t/t) \to 0$ as $t \to \infty$.

(c) Show that $A_t \to \infty$ as $t \to \infty$ almost surely.

**6** Let $M$ be a Poisson random measure on $(0,\infty)$ with intensity $\lambda dt$, where $\lambda \in (0,\infty)$. Let $(Y_n : n \in \mathbb{N})$ be a sequence of independent random variables, independent of $M$ and each uniformly distributed on $[0,1]$. Given a measurable function $g$ on $[0,1]$, define

$$X_t = X_t^g = \sum_{n=1}^{N_t} g(Y_n)$$

where $N_t = M(0,t]$.

(a) Show that $(X_t)_{t\geqslant 0}$ is a Lévy process.

(b) In the case where $g \geqslant 0$, show that

$$\mathbb{E}(X_t) = \lambda t \int_0^1 g(y)dy.$$

(c) Find a necessary and sufficient condition on $g$ for $(X_t)_{t\geqslant 0}$ to be a martingale.

(d) Given continuous functions $g_1$ and $g_2$ on $[0,1]$, find a necessary and sufficient condition on $g_1$ and $g_2$ for the processes $(X_t^{g_1})_{t\geqslant 0}$ and $(X_t^{g_2})_{t\geqslant 0}$ to be independent.

**END OF PAPER**