UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 1:30 pm to 4:30 pm

PAPER 150

ANALYTIC NUMBER THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question you may assume without proof Chebyshev's estimate $\psi(x) \asymp x$.

(a) Prove that

$$\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1).$$

(b) Prove that

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c + O(1/\log x)$$

for some constant c.

(c) Prove that if

$$\psi(x) = ax + (b + o(1))\frac{x}{\log x}$$

for some constants a and b then $a = 1$ and $b = 0$.

(d) Prove that

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^{-1} = C \log x + O(1)$$

for some constant C.

(e) Show that, for all $n \geqslant 3$,

$$\phi(n) \geqslant (C^{-1} + o(1))\frac{n}{\log\log n}$$

where ϕ is the Euler totient function and C is the constant from part (d).

2

(a) Define the *sifting function* $S(A, P; z)$, explaining all implicit notation used.

(b) State and sketch a proof of Selberg's upper bound for $S(A, P; z)$. You should include the construction of the sieve weights λ_d, but need not show that $|\lambda_d| \leqslant 1$.

(c) Let $h_1, \ldots, h_r$ be any distinct natural numbers. Show that

$$\#\{n \leqslant x \text{ such that } n + h_1, \ldots, n + h_r \text{ are all prime}\} \ll_{h_1,\ldots,h_r,r} \frac{x}{(\log x)^r}.$$

[You may use without proof the estimates

$$\sum_{\substack{n \leqslant x \\ (n,W)=1}} k^{\omega(n)} \gg_{k,W} x(\log x)^{k-1}$$

and $k^{\omega(n)} \ll_{k,\epsilon} n^{\epsilon}$, both valid for any integers $k, W \geqslant 1$ and $\epsilon > 0$.]

3 In this question $s = \sigma + it$ denotes a complex variable with $\Re(s) = \sigma$ and $\Im(s) = t$. You may assume without proof the Euler product identity

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } \sigma > 1.$$

(a) Show how the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ can be extended to a meromorphic function in the half-plane $\sigma > 0$.

(b) Show that $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ when $\sigma > 1$, where μ is the Möbius function.

(c) Show that there is an absolute constant $c > 0$ such that

$$\zeta(s) \neq 0 \text{ for } \sigma > 1 - \frac{c}{\log t} \text{ and } |t| \geqslant 4.$$

[You may assume that for $|t| \geqslant 4$ and $\frac{5}{6} \leqslant \sigma \leqslant 2$

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s - \rho} + O(\log|t|)$$

where the sum ranges over zeros ρ of $\zeta(s)$ such that $|\rho - (3/2 + it)| \leqslant 5/6$, and $-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$ uniformly for $8/9 \leqslant \sigma \leqslant 2$ and $|t| \leqslant 7/8$.]

(d) State Perron's formula, and use it to show that

$$\sum_{n \leqslant x} \mu(n) \ll x \exp(-c\sqrt{\log x})$$

for some absolute constant $c > 0$. [You may use without proof the estimates

$$\left|\frac{1}{\zeta(s)}\right| \ll \log|t| \text{ when } 1 - c_1/\log|t| < \sigma \text{ and } |t| \geqslant 7/8,$$

and

$$\left|\frac{1}{\zeta(s)}\right| \ll |s - 1| \text{ when } 1 - c_1/\log|t| < \sigma \leqslant 2 \text{ and } |t| \leqslant 7/8,$$

both valid for some absolute constant $c_1 > 0$.]

4 In this question χ is used to denote a Dirichlet character modulo q, and σ is used to denote the real part of the complex variable s. You may assume any standard properties of Dirichlet characters and Dirichlet series in the half-plane $\sigma > 1$.

(a) Prove that $L(1,\chi) \neq 0$ for any non-principal character χ. You may assume Landau's lemma concerning Dirichlet series with non-negative coefficients.

(b) Deduce that, for fixed a and q such that $(a,q) = 1$,

$$\psi(x;q,a) = \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} \Lambda(n) \to \infty \text{ as } x \to \infty.$$

(c) Describe the zero-free region for $L(s,\chi)$ in the strip $0 \leqslant \sigma \leqslant 1$, and explain what is meant by an 'exceptional zero'.

(d) Let a and q be such that $(a,q) = 1$. Suppose that q has an exceptional zero at β. State an asymptotic formula with error term for $\psi(x;q,a)$ valid for any x.

(e) Suppose that there exists some absolute $\epsilon > 0$ such that the bound

$$\psi(x;q,a) \leqslant (2-\epsilon)\frac{x}{\phi(q)} \text{ for } x \geqslant q^2$$

holds for all a and all q sufficiently large.

Show that this implies, for q sufficiently large, that an exceptional zero β for q must satisfy

$$\beta \leqslant 1 - \frac{c}{(\log q)^2}$$

for some constant $c > 0$ (which may depend on ϵ).

END OF PAPER