MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2019 1:30 pm to 3:30 pm

PAPER 149

INTRODUCTION TO APPROXIMATE GROUPS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State the Plünnecke–Ruzsa inequalities for a finite subset A of an abelian group satisfying $|A + A| \leqslant K|A|$.

(b) State Ruzsa's covering lemma and Ruzsa's triangle inequality for finite subsets of an arbitrary group. Show that if A is a finite symmetric subset of an arbitrary group such that $|A^3| \leqslant K|A|$ then $|A^m| \leqslant K^{m-2}|A|$ for every $m \geqslant 3$. Show, moreover, that A^2 is a K^3-approximate group. Give, with brief justification, an example to show that analogous results do not always hold if the assumption $|A^3| \leqslant K|A|$ is replaced with $|A^2| \leqslant K|A|$.

(c) Show that if A is a finite symmetric subset of an abelian group such that $|A+A| \leqslant K|A|$ then there exists a set X of size at most K^4 such that $mA \subset (m-2)X+2A$ for every $m \geqslant 3$. Deduce that $|mA| \leqslant Km^{K^4}|A|$ for every $m \in \mathbb{N}$. How does this compare to the bound on $|mA|$ given by the Plünnecke–Ruzsa inequalities as $m \to \infty$?

(d) Let G be an s-step nilpotent group in which every element has order at most r, and let $A \subset G$ be a finite K-approximate group. Show that there exists a subgroup H of G of size at most $r^{sK^s}|A|$ such that $A \subset H$. [You may assume the basic properties of nilpotent groups proved in the course, provided you state them clearly.]

2 Let $K \geqslant 2$. Let G be a torsion-free s-step nilpotent group, and let $A \subset G$ be a finite K-approximate group.

(a) Show that there exist $r \leqslant K^{O(1)}$ and $K^{O(1)}$-approximate groups $A_0, A_1 \ldots, A_r \subset A^{O(1)}$, each generating a subgroup of step less than s, such that $|A_0 \cdots A_r| \geqslant \exp(-K^{O(1)})|A|$. [You may assume any results from the course about sets of small doubling in abelian groups, and basic general properties of approximate groups, provided you state them clearly. Writing $\pi : G \to G/[G,G]$ for the quotient homomorphism, you may also assume that if a subgroup $H < G/[G,G]$ is finite or cyclic then $\pi^{-1}(H)$ has step less than s.]

(b) It turns out that by using more advanced results about sets of small doubling in abelian groups one can improve the bounds in the result of part (a) to $r \leqslant O(\log^{O(1)} 2K)$ and $|A_0 \cdots A_r| \geqslant \exp(-O(\log^{O(1)} 2K))|A|$ (here we write $\log^m x$ to mean $(\log x)^m$). In answering part (b) of this question you may assume these bounds in the result from part (a).

(i) Show that there exist $k \leqslant O(\log^{O(1)} 2K)$, a subset $X \subset A$ of size at most $\exp(O(\log^{O(1)} 2K))$, and $K^{O(1)}$-approximate groups $B_1 \ldots, B_k \subset A^{O(1)}$ such that each B_i generates a subgroup of step less than s and such that $A \subset XB_1 \cdots B_k$.

(ii) Deduce that there exist $m, n \leqslant O_s(\log^{O_s(1)} 2K)$, sets $X_1, \ldots, X_m \subset A^{O_s(1)}$ of size at most $\exp(O_s(\log^{O(1)} 2K))$, and $K^{O_s(1)}$-approximate groups $C_1, \ldots, C_n \subset A^{O_s(1)}$ such that each C_i generates an abelian subgroup and such that A is contained in the product, in some order, of the X_i and C_j.

(iii) Hence or otherwise show that there exists $q \leqslant O_s(\log^{O_s(1)} 2K)$ and $K^{O_s(1)}$-approximate groups $D_1, \ldots, D_q \subset A^{O_s(1)}$ such that each D_i generates an abelian subgroup, and such that $|D_1 \cdots D_q| \geqslant \exp(-O_s(\log^{O_s(1)} 2K))|A|$.

3 Given $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, define the map $f_{a,b} : \mathbb{C} \to \mathbb{C}$ via $f_{a,b}(z) = az + b$. Let $G = \{f_{a,b} : a \in \mathbb{C}\setminus\{0\},\ b \in \mathbb{C}\}$ be the group of these maps under composition. [You do not need to verify that G is a group.] Write $\pi : G \to G/[G,G]$ for the quotient homomorphism.

(a) Writing $\mathbb{C}^\times$ for the multiplicative group of non-zero complex numbers, show that $[G,G] \cong (\mathbb{C},+)$ and $G/[G,G] \cong \mathbb{C}^\times$. Write $\pi(f_{a,b}) \in \mathbb{C}^\times$ under this identification explicitly in terms of a and b. [*Hint: First check that* $f_{a,b} \circ f_{a',b'} = f_{aa',x}$ *for some* x *depending on* a, a', b, b', *and that* $[f_{a,1}, f_{1,b}] = f_{1,b(1-a^{-1})}$. *Then identify the set* $\{[f,g] : f,g \in G\}$ *of commutators in* G.]

(b) State Solymosi's sum-product theorem for finite sets $U, V, W \subset \mathbb{C}$.

(c) Let $K \geqslant 2$, and suppose that $A \subset G$ is a finite K-approximate group generating a non-abelian subgroup of G. Show that $|\pi(A)| \leqslant K^{O(1)}$. [*Hint: First check that* $f_{a,b} \circ f_{1,c} \circ f_{a,b}^{-1} = f_{1,ac}$.]

(d) Deduce that if $A \subset G$ is a finite K-approximate group generating a non-abelian subgroup of G then there is an abelian progression P of rank at most $K^{O(1)}$ and size at most $\exp(K^{O(1)})|A|$ such that A is covered by at most $K^{O(1)}$ left-translates of P. [*Hint: First show that* A *is contained in the union of at most* $K^{O(1)}$ *left-cosets of an abelian subgroup.*]

Provided you state them clearly, you may use without proof basic general properties of approximate groups that we proved in the course. You may also use the Freiman–Green–Ruzsa theorem without proof.

4 (a) Let $d > 0$. Show that there exists $N \in \mathbb{N}$ depending only on d such that the following holds. If G is a group with a finite symmetric generating set S containing the identity, and there is some $n \geqslant N$ such that $|S^n| \leqslant n^d|S|$, then there exist subgroups $H \lhd C < G$ with $H \subset S^{\lfloor n/2 \rfloor}$ and $[G : C] \leqslant O_d(1)$ such that C/H is nilpotent of step at most $O_d(1)$. [You may use any results from the course about approximate groups and sets of small doubling or tripling, provided you state them clearly.]

(b) Let $d > 0$. Show that there exists $N \in \mathbb{N}$ depending only on d such that the following holds. If G is a group with a finite symmetric generating set S containing the identity, and there is some $n \geqslant N$ such that $|S^n| \leqslant n^d|S|$, then G has an $O_d(1)$-step nilpotent subgroup of finite index.

(c) If G is a finite group, and S is a symmetric generating set for G containing the identity, then the *diameter* of G with respect to S is written $\mathrm{diam}_S(G)$ and defined via $\mathrm{diam}_S(G) = \min\{n : S^n = G\}$.

(i) Show that $\mathrm{diam}_S(G) < |G|$ for arbitrary finite G and S.

(ii) A group G is said to be *simple* if its only normal subgroups are $\{1\}$ and G. Show that for every $\varepsilon > 0$ there exists $\lambda > 0$ depending only on ε such that if G is a non-abelian finite simple group with a finite symmetric generating set S containing the identity then $\mathrm{diam}_S(G) \leqslant \max\{|G|^{\varepsilon}, \lambda\}$. [*Hint: Part (a) may be helpful.*]

END OF PAPER