UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2019 1:30 pm to 4:30 pm

PAPER 148

ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What does it mean for a commutative ring T to be integral over a subring R?

Suppose T is integral over R. Let π be the canonical map from $\operatorname{Spec} T$ to $\operatorname{Spec} R$. Show that π is surjective. Let $P \in \operatorname{Spec} R$. Show that $\pi^{-1}(P)$ cannot contain prime ideals Q_1 and Q_2 with Q_1 strictly contained in Q_2.

Define the dimension $\dim T$ of T. Prove that $\dim R = \dim T$.

Let k be a field of characteristic 0. Let $T = k[X,Y,Z]/(XY + YZ + ZX)$. Find a subring R of T such that T is integral over R and R is isomorphic to a polynomial algebra over k.

2

Let R be a commutative Noetherian ring. Let I be an ideal, and let M be a finitely generated R-module.

Let $S = 1 + I$. Define the ring $S^{-1}R$, the module $S^{-1}M$ and the canonical maps $R \longrightarrow S^{-1}R$ and $M \longrightarrow S^{-1}M$. Show that $S^{-1}R$ and $S^{-1}M$ are Noetherian. What is the relationship between $\operatorname{Spec} R$ and $\operatorname{Spec} S^{-1}R$? Justify your answer.

Show that the kernel of the map $M \longrightarrow S^{-1}M$ is $\bigcap_{j=1}^{\infty} I^j M$. [*You may use the Artin-Rees Lemma if correctly stated.*]

Give an example of a non-Noetherian ring R and an ideal I such that the kernel of $R \longrightarrow S^{-1}R$ is not equal to $\bigcap_{j=1}^{\infty} I^j$.

3

Let R be a commutative Noetherian ring. Define what is meant by the height of a prime ideal P.

State and prove Krull's principal ideal theorem.

Show that an integral domain is a unique factorisation domain if and only if all height one primes are principal.

4

Let R be a commutative Artinian ring. Show that there are only finitely many prime ideals of R and that they are all maximal.

Define what is meant by the nilradical of R and show that it is a nilpotent ideal. Show that R is Noetherian.

Now suppose that R has a unique maximal ideal P. Suppose that P/P^2 is a one dimensional vector space over the field R/P. Show that all ideals of R are principal.

Let k be a field, and let T be a semisimple, not necessarily commutative, k-algebra which is finite dimensional as a k-vector space. State and prove the Artin-Wedderburn theorem for T. [*You may assume that T is completely reducible as a right T-module.*]

5

Let k be a field, and let R be a k-algebra. Let M be an $R-R$-bimodule.

Define the Hochschild chain complex for R, and define the Hochschild homology groups $HH_j(R,M)$ and cohomology groups $HH^j(R,M)$.

What is meant by a derivation D from R to M? When is such a derivation inner? Denote the space of derivations by $\mathrm{Der}(R,M)$. Describe $HH^1(R,M)$ in terms of derivations.

Let Ω be the kernel of the k-linear map from $R \otimes_k R$ to R sending $r_1 \otimes r_2$ to $r_1 r_2$. Show that there is a derivation from R to Ω such that the map from $\mathrm{Hom}_{R-R}(\Omega, M)$ to $\mathrm{Der}(R,M)$ sending θ to the composition of D with θ is an isomorphism. Which $\theta \in \mathrm{Hom}_{R-R}(\Omega, M)$ correspond to inner derivations?

6

Let k be a field and S be a commutative Noetherian graded $k-$algebra. State and prove the Hilbert-Serre theorem concerning the Poincare series of a finitely generated graded S-module V.

Let R be a commutative k-algebra generated by $x_1, \ldots, x_n$. Set R_j to be the k-span of the polynomial expressions in the x_i of total degree less than or equal to j.

What can be said about $\dim_k R_j$? Justify your answer. Illustrate your answer with the example $k[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}]$ with generating set $Y_1, Y_1^{-1}, Y_2, Y_2^{-1}$.

END OF PAPER