UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2019 9:00 am to 11:00 am

PAPER 147

INTRODUCTION TO DISCRETE ANALYSIS

Answer Question 1 and two further questions

*There are **FOUR** questions in total*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let G be a bipartite graph with finite vertex sets X and Y and with at least $\delta|X||Y|$ edges. Prove that the number of ordered quadruples $(x_1, x_2, y_1, y_2) \in X \times X \times Y \times Y$ such that all four pairs $x_i y_j$ are edges of G is at least $\delta^4 |X|^2 |Y|^2$.

(ii) Let $A \subset \mathbb{Z}_N$. Let $(x_1, x_2, \ldots, x_6) \in \mathbb{Z}_N^6$ be chosen uniformly at random from the set of all sextuples such that $x_1 + x_2 + x_3 = x_4 + x_5 + x_6$. Prove that the probability that every x_i is an element of A is $\sum_r |\hat{A}(r)|^6$, where $\hat{A}$ is the Fourier transform of the characteristic function of A. You may assume facts and definitions of Fourier transforms.

(iii) Let P be a polynomial in n variables with coefficients in $\mathbb{F}_p$. Show that if P has total degree less than p and is not the zero polynomial, then there exists an n-tuple $(a_1, \ldots, a_n) \in \mathbb{F}_p^n$ such that $P(a_1, \ldots, a_n) \neq 0$.

(iv) Let $f : \mathbb{Z}_N \to \mathbb{C}$ be a function such that $\|f\|_\infty \leqslant 1$ and such that

$$|\mathbb{E}_{x,a,b} f(x) \overline{f(x-a) f(x-b)} f(x-a-b) \omega^{-2ab}| \geqslant c.$$

Prove that there exists $r \in \mathbb{Z}_N$ such that

$$|\mathbb{E}_x f(x) \omega^{-rx-x^2}| \geqslant c^{1/2}.$$

Here, $\omega = \exp(2\pi i/N)$ and the variables in the averages range independently over $\mathbb{Z}_N$. You may assume facts and definitions of Fourier transforms.

2 Let A be a subset of $\{1, 2, \ldots, N\}$ of density $\delta > 0$ that contains no arithmetic progression of length 3. Prove that if N is sufficiently large, then there is an arithmetic progression $P \subset \{1, 2, \ldots, N\}$ of length at least $\eta\sqrt{N}$ such that $|A \cap P| \geqslant (\delta + c\delta^2)|P|$, where $\eta > 0$ depends on δ only and $c > 0$ is an absolute constant. (Definitions and basic facts concerning the discrete Fourier transform may be assumed.)

3 (i) Let G be a bipartite graph with finite vertex sets X and Y. Let $A \subset X$ and $B \subset Y$ and let $\epsilon > 0$. What does it mean to say that the pair (A, B) is ϵ-regular? If X is partitioned into sets $X_1 \cup \cdots \cup X_r$ and Y into sets $Y_1 \cup \cdots \cup Y_s$, what does it mean to say that the pair of partitions is ϵ-regular?

(ii) Prove that for every $\epsilon > 0$ there exists K such that for every bipartite graph G with finite vertex sets X and Y there is an ϵ-regular pair of partitions of X and Y with at most K cells in each partition. (You may assume basic facts about conditional expectations.)

4 Prove that there exists a constant $C < 3$ such that for every positive integer n, every subset of $\mathbb{F}_3^n$ of size at least C^n contains three distinct elements x, y, z such that $x+y+z = 0$. (You may assume that if $X_1, \ldots, X_n$ are independent random variables, each uniformly distributed in the set $\{-1, 0, 1\}$, then the probability that $\sum_i X_i \geqslant n/3$ is at most $\exp(-n/12)$.)

END OF PAPER