UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 1:30 pm to 4:30 pm

PAPER 146

SYMPLECTIC TOPOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let (X, ω_x, J) be a symplectic manifold with a compatible almost complex structure J. Let $(\Sigma, \omega_\Sigma, j)$ be a compact Riemann surface. For any smooth map $u : \Sigma \to X$, define the energy of u. Define what it means for u to be J-holomorphic, and show that J-holomorphic curves minimise energy.

Give an example of a monotonicity theorem for J-holomorphic curves, and state the Gromov non-squeezing theorem.

Fix $\epsilon \ll 1$. For any $R > 0$, find a linear Lagrangian subspace of $\mathbb{R}^4 \cap B(R)$ whose ϵ neighbourhood projects injectively to $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2$, equipped with the quotient symplectic form. Hence or otherwise construct a symplectic embedding $B(r) \hookrightarrow \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2$ for arbitrarily large r.

2

State the neighbourhood theorem for a closed symplectic submanifold of a symplectic manifold.

Let C be a smooth surface of genus g. Prove that if symplectic 4-manifolds X and Y contain embedded symplectic submanifolds diffeomorphic to C and of self-intersection zero, then the fibre sum $X \#_C Y$ carries a natural symplectic structure.

Construct a closed, compact symplectic 4-manifold with fundamental group

$$\langle a, b, c \,|\, ba = ac \rangle,$$

justifying your answer. You may use any property of rational elliptic surfaces that you require, provided it is clearly stated.

How about a closed compact symplectic manifold with the same fundamental group and dimension 100?

3 Write down an expression for the standard Kähler form on $\mathbb{P}^n$, normalised so that a line has symplectic area π, on a coordinate patch. State Darboux' theorem. Carefully define the blow-up $\tilde{X}_\lambda$ of a symplectic manifold X at a point p with parameter λ. Can you give an example where $\tilde{X}_\lambda$ and $\tilde{X}_{\lambda'}$ are not symplectomorphic for some $\lambda \neq \lambda'$?

Show that for all $r < 1$, $B^4(r)$ symplectically embeds into $\mathbb{P}^1 \times \mathbb{P}^1$, equipped with the product of the standard Kähler form with itself. Is this bound sharp? Justify your answer; you may assume standard properties of J-holomorphic curves provided you state them clearly.

4 State and prove Darboux' theorem. Define the notion of a Hamiltonian vector field on a closed symplectic manifold (X, ω), and show that its flow preserves the symplectic form. Hence or otherwise, show that symplectomorphisms of X act transitively on points.

Let $\mathbb{R}^{2n}$ be equipped with the standard symplectic form. Show that there exists a pair of compact, connected Lagrangian submanifolds $L_1, L_2 \subset \mathbb{R}^2$ such that no symplectomorphism of $\mathbb{R}^2$ maps L_1 to L_2. Does there exist such a pair of compact connected Lagrangians in $\mathbb{R}^{2n}$ for general n? Justify your answer.

END OF PAPER