UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2019 1:30 pm to 4:30 pm

PAPER 145

IWASAWA ALGEBRAS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give the definition of a *filtration* ω on a group G. When is such a filtration ω a *p-valuation*?

Suppose p is an odd prime. Show that the kernel of the natural group homomorphism $GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{Z}_p/p\mathbb{Z}_p)$ can be given a p-valuation.

Show that if ω_1 and ω_2 are p-valuations on a group G then

$$\omega(g) = \min(\omega_1(g), \omega_2(g)) \text{ for } g \in G$$

defines a p-valuation on G.

2

What is the *associated graded group* $gr\, G$ of a separated filtered group (G, ω)?

Explain how to use the commutator bracket on G to define a graded Lie algebra structure on $gr\, G$ that is non-abelian if there are non-identity elements $x, y \in G$ with $\omega(x^{-1}y^{-1}xy) = \omega(x) + \omega(y)$.

Show that if ω is a p-valuation then $gr\, G$ may be viewed as a graded $\mathbb{F}_p[t]$-Lie algebra where $\mathbb{F}_p[t]$ is given its natural grading with $\deg t = 1$.

Compute the Lie structure on $gr\, G$ when

$$G = \left\{ \begin{pmatrix} 1+a & b \\ 0 & 1 \end{pmatrix} \middle| \, a, b \in p\mathbb{Z}_p \right\}$$

and

$$\omega\left(\begin{pmatrix} 1+a & b \\ 0 & 1 \end{pmatrix} \right) = \min(v_p(a), v_p(b)).$$

[*Throughout this question you may assume the Hall-Petrescu Formula and any basic commutator identities hold without proof provided you state them clearly.*]

3 Suppose that (G, ω) is a complete p-valued group of finite rank. Show that the group algebra $\mathbb{Z}_p[G]$ may be equipped with a filtration such that $\mathbb{Z}_p[G]_\lambda$ is the $\mathbb{Z}_p$-module spanned by the set

$$X_\lambda = \left\{ p^r(g_1 - 1)\cdots(g_s - 1) \;\middle|\; r + \sum_{i=1}^{s} \omega(g_i) \geqslant \lambda \text{ for } g_1, \ldots, g_s \in G \right\}$$

for each $\lambda \geqslant 0$. Prove that with respect to this filtration $gr\,\mathbb{Z}_p[G]$ is naturally a graded $\mathbb{F}_p[t]$-algebra and there is a surjective graded $\mathbb{F}_p[t]$-algebra homomorphism

$$U_{\mathbb{F}_p[t]}(gr\,G) \to gr(\mathbb{Z}_p[G]).$$

Suppose now that p is odd and G is the kernel of the natural homomorphism $SL_2(\mathbb{Z}_p) \to SL_2(\mathbb{Z}_p/p\mathbb{Z}_p)$ equipped with the p-valuation coming from the natural p-adic filtration on $M_2(\mathbb{Z}_p)$. By considering the ordered basis

$$\left(E = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix}, F = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \right)$$

for G and writing $e = E - 1$, $f = F - 1$ and $h = H - 1$ in $\mathbb{Z}_p[G]$ show that

$$\begin{aligned} ef - fe + \mathbb{Z}_p[G]_3 &= ph + \mathbb{Z}_p[G]_3, \\ he - eh + \mathbb{Z}_p[G]_3 &= 2pe + \mathbb{Z}_p[G]_3 \text{ and} \\ hf - fh + \mathbb{Z}_p[G]_3 &= -2pf + \mathbb{Z}_p[G]_3. \end{aligned}$$

4 Suppose that G is a complete p-valued group of finite rank with centre Z. Prove that the centre of the completed group algebra $\mathbb{F}_pG$ is isomorphic to the completed group algebra $\mathbb{F}_pZ$. Thus compute the centre of $\mathbb{F}_pG$ when G is the kernel of the natural homomorphism $GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{Z}_p/p\mathbb{Z}_p)$.

5 What is the definition of a $\mathbb{Q}_p$-*Banach algebra*? What is required for a p-valued group (G, ω) to be p-*saturated*?

Suppose that (G, ω) is a p-saturated p-valued group of finite rank. Explain why the usual power series expansion for $\log(1+T)$ induces a function from G to the $\mathbb{Q}_p$-Banach algebra $\widehat{\mathbb{Q}_p G}$. Prove that the image of this function is a $\mathbb{Z}_p$-Lie subalgebra of $\widehat{\mathbb{Q}_p G}$ with its usual commutator bracket.

Show moreover that if $(g_1, \ldots, g_d)$ is an ordered basis for G then $\log(g_1), \ldots, \log(g_d)$ is a basis for $\mathcal{P}(\widehat{\mathbb{Q}_p G})$ as a $\mathbb{Q}_p$-vector space.

[You may assume standard properties of the usual filtration on $\widehat{\mathbb{Q}_p G}$, the Hausdorff series and the commutator Campbell–Baker–Hausdorff series.]

Describe $\log(G)$ when

$$G = \left\{ \begin{pmatrix} 1+a & b \\ 0 & 1 \end{pmatrix} \middle| \, a, b \in p\mathbb{Z}_p \right\}$$

and

$$\omega\left(\begin{pmatrix} 1+a & b \\ 0 & 1 \end{pmatrix} \right) = \min(v_p(a), v_p(b)).$$

END OF PAPER