UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 9:00 am to 11:00 am

PAPER 144

MODEL THEORY

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State and prove the Tarski-Vaught test.

(b) Let T be a complete theory with a monster model $\mathcal{U}$. Let $\mathcal{M}$ be a model (that is, $\mathcal{M}$ is small and $\mathcal{M} \preceq \mathcal{U}$) and let $\varphi(x, z)$ be an L-formula. Suppose there are finitely many sets of the form $\varphi(a, \mathcal{U})$, where $a \in \mathcal{U}$. Prove that all these sets are definable over $\mathcal{M}$.

Suppose conversely that for all $a \in \mathcal{U}$ there is $b \in \mathcal{M}$ such that $\varphi(a, \mathcal{U}) = \varphi(b, \mathcal{U})$. Does it follow that there are only finitely many sets of the form $\varphi(a, \mathcal{U})$, where $a \in \mathcal{U}$?

2 Recall that the theory T_{rg} is the theory of random graphs in the language $L_{\text{gph}} = \{R\}$, where R is a binary relation symbol.

(a) Prove that any finite partial embedding between two countable models of T_{rg} extends to an isomorphism.

(b) Let $\mathcal{M} \models T_{\text{rg}}$ be countable, and let $P_1, \ldots, P_n$ be a partition of M. Prove that $\mathcal{M}$ is isomorphic to $\mathcal{P}_i$ for some i, where $\mathcal{P}_i$ is the structure induced by $\mathcal{M}$ on P_i.

3 Let T be a complete theory with a monster model $\mathcal{U}$ and let $A \subseteq \mathcal{U}$ be a (small) subset.

(a) Prove that the following are equivalent:

(i) $a \in \operatorname{acl}(A)$;

(ii) $a \in \mathcal{M}$ for every model $\mathcal{M}$ containing A.

(b) Let $\mathcal{N}$ be a model. Prove that for every $A \subseteq \mathcal{N}$ there is a model $\mathcal{M}$ such that

$$\operatorname{acl}(A) = \mathcal{M} \cap \mathcal{N}.$$

[*You may assume the following result: if C is a finite set such that $C \cap M \neq \emptyset$ for every model $\mathcal{M}$ containing A, then $C \cap \operatorname{acl}(A) \neq \emptyset$.*]

4 Let T be a strongly minimal L-theory. Work in the monster model $\mathcal{U}$ of T.

(a) Let $f : \mathcal{U} \to \mathcal{U}$ be an elementary map. Let $b \notin \mathrm{acl}(\mathrm{dom}(f))$ and $c \notin \mathrm{acl}(\mathrm{ran}(f))$. Show that $f \cup \{\langle b, c\rangle\}$ is elementary. Hence prove that if two models have the same dimension then they are isomorphic. [You may assume standard properties of algebraic closure.]

(b) Show that for every $\mathcal{N} \preceq \mathcal{U}$ such that $|\mathcal{N}| \geqslant |L| + \omega$ the following are equivalent:

(i) $\mathcal{N}$ is saturated;

(ii) $\dim(\mathcal{N}) = |\mathcal{N}|$.

[*Hint: For (i) $\Rightarrow$ (ii) consider a suitable type over a basis of* $\mathcal{N}$*. For (ii) $\Rightarrow$ (i) use a suitable characterisation of saturation.*]

END OF PAPER