UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 9:00 am to 11:00 am

PAPER 143

INTRODUCTION TO GEOMETRIC GROUP THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove the Nielsen–Schreier formula for the rank of a finite-index subgroup H of a free group F_n on n generators. [You may state and use results on covering spaces without proof.]

(b) The *rank* of a (not necessarily free) group is defined to be the minimal cardinality of a generating set. Give an example to show that the Nielsen–Schreier formula no longer holds for groups that are not free.

(c) Prove that one can replace the equality in the Nielsen–Schreier formula with an inequality so that the resulting inequality holds for the rank of a subgroup of any finitely generated group.

(d) Prove that the number of subgroups of a given index $n \in \mathbb{N}$ in a finitely generated group G is finite.

2

(a) Define the *Cayley graph* of a finitely generated group G with respect to a finite generating set S. Give an example of two non-isomorphic **infinite** finitely generated groups G and H and finite generating sets S and T for G and H respectively that give rise to the same Cayley graph.

(b) Define what it means for two metric spaces to be *quasi-isometric*. State the result of Švarc–Milnor. Show that for a finitely generated group, different choices of finite generating set give rise to quasi-isometric Cayley graphs.

(c) Let G and H be finitely generated groups and let $\varphi : G \to H$ be a (not necessarily surjective) homomorphism. Show that φ is a quasi-isometry if and only if $\ker(\varphi)$ and $H/\varphi(G)$ are both finite.

(d) Can a non-trivial finitely generated group G be quasi-isometric to the free product $G * G$? Give either a proof that this is not possible, or an example with a proof that it satisfies the condition.

3

(a) Define the *growth function* β_{X,x_0} of a discrete metric space X with respect to the basepoint x_0. Prove that for a given Cayley graph of a finitely generated group, the growth function is independent of the choice of basepoint.

(b) Let G be a finitely generated group, and let S be a finite generating set of G. Recall that $\beta_{G,S}$ denotes the growth function of the Cayley graph of G with respect to S. Let $H \leqslant G$ be a finitely generated subgroup of the finitely generated group G. Show that there exist finite generating sets S and T of G and H respectively such that $\beta_{H,T}(r) \leqslant \beta_{G,S}(r)$ for all $r \geqslant 0$.

(c) Define what it means for a finitely generated group to have *polynomial growth*. Let G be a finitely generated, 2-step nilpotent group, i.e. $G \geqslant [G,G] \geqslant [[G,G],G] = \{1\}$. Show that G has polynomial growth. [You may assume that for such a G, the commutator subgroup $[G,G]$ can be generated by commutators of generators (and inverses of generators) of G, and that finitely generated abelian groups have polynomial growth.]

(d) Show that the group $SL_2(\mathbb{Z})$ has exponential growth.

4

(a) Define what it means for a finitely generated group to be *amenable* in terms of measure. For a group G acting on a set X, define what it means for two subsets $A, B \subseteq X$ to be G-*equidecomposable*, and define what it means for X to be G-*paradoxical*. Show that the free group F_2 is not amenable.

(b) Define the *Følner condition* for a finitely generated group. Show that if a finitely generated group G satisfies the Følner condition, then it is amenable. [You may assume relevant results from topology.]

(c) Verify that $\mathbb{Z}$ satisfies the Følner condition.

(d) Can a group of the form $G * G$ be amenable, for G a non-trivial finitely generated group? Justify your answer.

END OF PAPER