MAT3, MAMA

## MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 1:30 pm to 3:30 pm

# PAPER 142

# CHARACTERISTIC CLASSES AND K-THEORY

*Attempt no more than **TWO** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** For a space $X$, define the ring $K^0(X)$.

Let $\pi_i : E_i \to X$, $i = 0, 1$, be $d$-dimensional complex vector bundles over a finite CW-complex $X$ of dimension $k$ such that $E_0 \oplus \underline{\mathbb{C}}_X \cong E_1 \oplus \underline{\mathbb{C}}_X$. If $k \leqslant 2d$, prove that $E_0 \cong E_1$. *[You may use any results about the classification of vector bundles provided they are clearly stated.]*

Prove that two $d$-dimensional complex vector bundles over $\mathbb{CP}^d$ having the same Chern classes are isomorphic. *[You may use any description of the K-theory of $\mathbb{CP}^d$, and any properties of the Chern character provided they are clearly stated.]*

**2** What are the *elementary symmetric polynomials* $e_i$? What are the *power sum polynomials* $p_i$? Prove the identity

$$p_n - e_1 p_{n-1} + e_2 p_{n-2} - \cdots + (-1)^{n-1} e_{n-1} p_1 + n(-1)^n e_n = 0.$$

If $\beta$ denotes the Bockstein operation associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

show that $\beta(w_1(L)) = w_1(L)^2$ for any real line bundle $p : L \to X$ over a compact Hausdorff space $X$. Hence show that for any real vector bundle $\pi : E \to X$ over a compact Hausdorff space $X$ the identity

$$\beta(w_j(E)) = w_1(E) w_j(E) + (j+1) w_{j+1}(E)$$

holds. *[You may use that $\beta(ab) = \beta(a)b + a\beta(b)$, and any results from the course provided that they are clearly stated.]*

**3** What is the $K$-theory *Euler class* $e^K(E)$ of a complex vector bundle $\pi : E \to X$? What is the $K$-theory *Gysin sequence* associated to this vector bundle?

By constructing an appropriate vector bundle isomorphism, identify the $K$-theory class of the tangent bundle $T\mathbb{CP}^n$ in terms of the tautological line bundle $\gamma_{\mathbb{C}}^{1,n+1}$. Hence determine the $K$-theory Euler class $e^K(T\mathbb{CP}^n) \in K^0(\mathbb{CP}^n)$. *[You may use any description of the K-theory of $\mathbb{CP}^n$ without proof.]*

Compute the ring $K^0(\mathbb{S}(T\mathbb{CP}^n))$ and the abelian group $K^{-1}(\mathbb{S}(T\mathbb{CP}^n))$.

**4** Define the Adams operations, and show that they give ring homomorphisms

$$\psi^\ell : K^0(X) \longrightarrow K^0(X)$$

which satisfy $\psi^\ell(L) = L^\ell$ when $\pi : L \to X$ is a complex line bundle. *[You may use the splitting principle and any results from the theory of symmetric polynomials provided they are clearly stated.]*

Let $\mathbb{CP}_k^{n+k} = \mathbb{CP}^{n+k}/\mathbb{CP}^{k-1}$ denote the truncated projective space,

$$i : S^{2k} = \mathbb{CP}_k^k \longrightarrow \mathbb{CP}_k^{n+k}$$

denote the inclusion, and suppose that $i$ has a retraction $r : \mathbb{CP}_k^{n+k} \to S^{2k}$. By considering the Adams operation $\psi^2$ show that if $n = 2$ then $k$ is divisible by 24.

***END OF PAPER***