UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2019 1:30 pm to 4:30 pm

PAPER 141

3-MANIFOLDS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

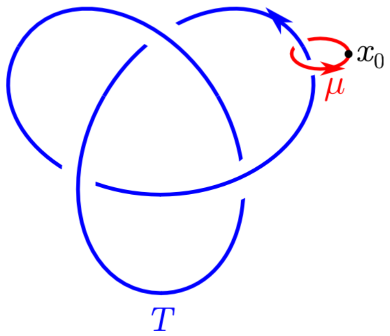
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The questions below refer to the following oriented link diagram (T, μ) for the trefoil knot $T \subset S^3$, with exterior $X_T := S^3 \setminus \nu(T)$, together with the shown choice of meridian $\mu \subset X_T$ for T. You may use any results from lecture or example sheets.



(a) Draw and label a Dehn presentation Heegaard diagram $\mathcal{H} := (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, x_0)$ for X_T, as specified by the above knot diagram for T.

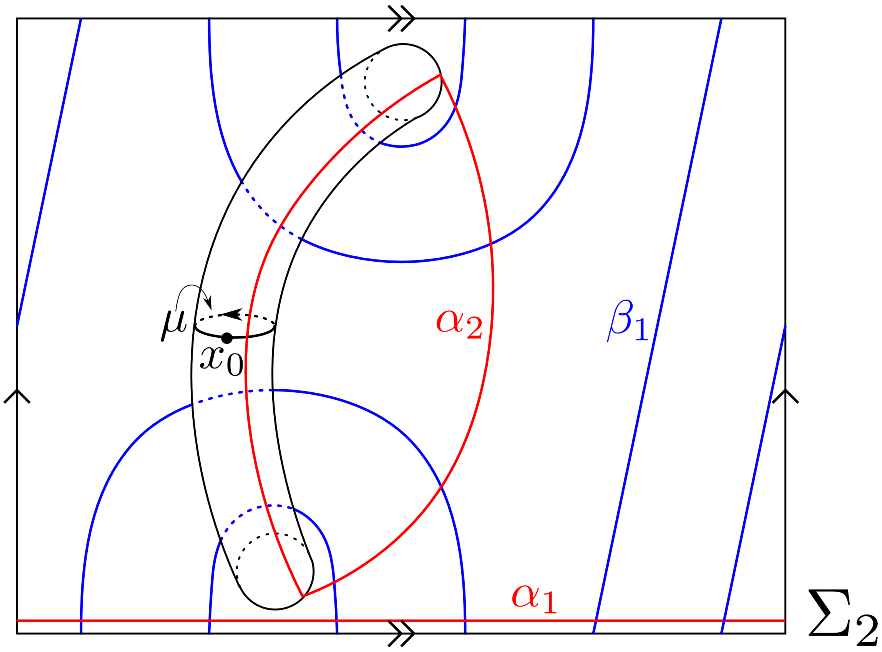
(b) Write down the (Dehn) presentation of $\pi_1(X_T, x_0)$ associated to this Heegaard diagram, briefly explaining the interpretation of the presentation's generators and relators with respect to the Heegaard diagram.

(c) Compute the images of the generators of this presentation under the abelianisation map $\varphi : \pi_1(X_T, x_0) \to H_1(X_T; \mathbb{Z})$, expressing your answers in terms of $\varphi(\mu)$.

(d) For the presentation computed in part (b), compute the Alexander matrix A of Fox derivatives of relators with respect to generators (before taking minors). Express your answer in terms of $t \in \mathbb{Z}[t^{-1}, t] \simeq \mathbb{Z}[H_1(X_T; \mathbb{Z})]$, for $t := [\varphi(\mu)]$ the expression of $\varphi(\mu)$ in multiplicative notation.

(e) Drawing a copy of the above knot diagram for T, label *one* Kauffman state of your choice, and circle the entries of the Alexander matrix A whose signed product gives the summand of the Alexander polynomial corresponding to your chosen Kauffman state.

2 The following questions refer to the Heegaard diagram $\mathcal{H} = (\Sigma_2, (\alpha_1, \alpha_2), (\beta_1), x_0)$ shown below, specifying a 3-manifold $M_{\mathcal{H}}$ with torus boundary $\partial M_{\mathcal{H}} = \mathbb{T}^2$. The genus-2 surface Σ_2 is presented as the fundamental domain of a twice punctured torus with annulus attached, oriented by the right-hand rule. The curve μ is not part of the data of $\mathcal{H}$.



You may use any results from lecture or example sheets for the following questions.

(a) Write down a presentation for the fundamental group $\pi_1(M_{\mathcal{H}}, x_0)$ of $M_{\mathcal{H}}$, and compute the image of its generators under the abelianisation homomorphism $\pi_1(M_{\mathcal{H}}, x_0) \to H_1(M_{\mathcal{H}}; \mathbb{Z})$. You do not need to justify either computation.

(b) Show that the Dehn filling $M_{\mathcal{H}}(\mu)$ of slope μ of $M_{\mathcal{H}}$ is homeomorphic to a lens space.

(c) Give two examples of Seifert fibered spaces, with different numbers of exceptional fibers, that are each homeomorphic to $M_{\mathcal{H}}(\mu)$. [There are many possible answers.]

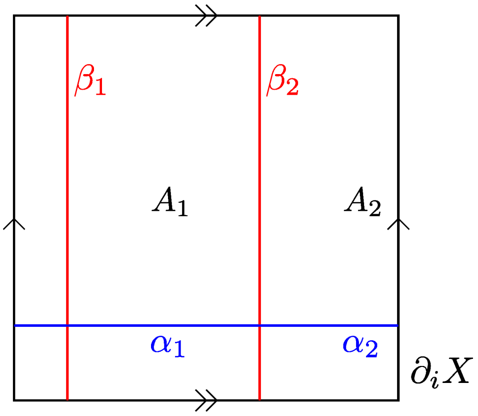
Let $K = \text{core}(M_{\mathcal{H}}(\mu) \setminus M_{\mathcal{H}}) \subset M_{\mathcal{H}}(\mu)$ be the core knot of the Dehn filling $M_{\mathcal{H}}(\mu)$. From now on, you may assume that K is a fibered knot. That is, there is a fibration $\pi : M_{\mathcal{H}} \to S^1$ such that K is isotopic to the boundary of any fiber of π.

(d) Specify a compact surface Σ such that $M_{\mathcal{H}}$ is homeomorphic to the mapping torus M_{φ}^t of some homeomorphism $\varphi : \Sigma \to \Sigma$. [Hint: how would you determine its genus?] In addition, specify a representative up to conjugacy for the induced homomorphism $\varphi_* : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})$. Again, you do not need to prove any results already proved in lecture or example sheets.

(e) According to the Nielsen-Thurston classification theorem for elements of the mapping class group of an oriented surface with boundary or punctures, is $M_{\mathcal{H}}$ Seifert fibered, a manifold with hyperbolic interior, or neither?

3 (a) Let X_0 and X_1 be closed oriented n-manifolds. Show that there is an $n+1$-dimensional 1-handle-attachment cobordism from $-(X_0 \amalg X_1)$ to the connected sum $X_0 \# X_1$. (A brief, informal explanation is sufficient.)

(b) Let X be a compact oriented 3-manifold with a toroidal boundary component $\partial_i X \cong \mathbb{T}^2$, and let $\alpha \subset \partial_i X$ be an embedded closed curve. Suppose $\beta_1, \beta_2 \subset \partial_i X$ are mutually-isotopic disjointly-embedded closed curves such that $|\alpha \cap \beta_1| = |\alpha \cap \beta_2| = 1$. This implies $\beta_1 \amalg \beta_2 \subset \partial_i X$ splits $\partial_i X$ into two open annuli $\mathring{A}_1 \amalg \mathring{A}_2 = \partial_i X \setminus (\beta_1 \amalg \beta_2)$ with respective closures $A_1 = \beta_1 \amalg \mathring{A}_1 \amalg \beta_2$ and $A_2 = \beta_2 \amalg \mathring{A}_2 \amalg \beta_1$. Likewise, β_1 and β_2 cut α into two embedded arcs $\alpha_1 := \alpha \cap A_1$ and $\alpha_2 := \alpha \cap A_2$, as in the figure below.



Let $f : A_1 \to A_2$ be an orientation-reversing homeomorphism restricting to the identity on $\beta_1 \amalg \beta_2$ and sending α_1 to α_2. Show that the Dehn filling $X(\alpha)$ is homeomorphic to the quotient $X/(A_1 \sim A_2)$ identifying $x \in A_1$ with $f(x) \in A_2$. [You may use the fact that boundaries have collar neighborhoods.]

For parts (c) and (d), let $K_1 \subset Y_1$ and $K_2 \subset Y_2$ be oriented knots in compact oriented 3-manifolds Y_1 and Y_2 with exteriors $X_1 := Y_1 \setminus \nu(K_1)$ and $X_2 := Y_2 \setminus \nu(K_2)$. Let $K_1 \# K_2 \subset Y_1 \# Y_2$ be the connected sum of $K_1 \subset Y_1$ and $K_2 \subset Y_2$, with exterior $X_\# := Y_1 \# Y_2 \setminus \nu(K_1 \# K_2)$.

(c) Describe a longitude for $K_1 \# K_2$ in terms of λ_1 and λ_2.

(d) Let $Y_1' := X_1(\lambda_1)$ be the integer Dehn surgery of slope λ_1 on $K_1 \subset Y_1$, and let $K_1' := \mathrm{core}(Y_1' \setminus X_1) \subset Y_1'$ be the resulting core knot, but oriented to have longitude μ_1, where μ_1 is the meridian of K_1 oriented so that $\mu_1 \cdot \lambda_1 = 1$. Show that there is an integer Dehn surgery $Y_\#'$ of $Y_1 \# Y_2$ along $K_1 \# K_2 \subset Y_1 \# Y_2$, such that $Y_\#'$ is homeomorphic to the splice Y of Y_1' and Y_2 along $K_1' \subset Y_1'$ and $K_2 \subset Y_2$. [Recall that a splice is a toroidal union of knot exteriors exchanging meridians with longitudes.] You may suppress inclusion maps from notation if desired.

4 Let $K_C \subset S^3$ be an oriented knot with exterior $X_C := S^3 \setminus \nu(K_C)$, and let $U \amalg K_P \subset S^3$ be a two-component oriented link with component exteriors $X_U := S^3 \setminus \nu(U)$ and $X_P := S^3 \setminus \nu(K_P)$, where $U \subset S^3$ is the unknot, such that $[U] \neq 0 \in H_1(X_P; \mathbb{Z})$. Let X_C^P denote the splice of S^3 and X_P along $K_C \subset S^3$ and $U \subset X_P$. [Recall that a splice is a toroidal union of knot exteriors exchanging meridians with longitudes.] For the following questions, you may use any properties of the Turaev torsion from lectures.

(a) Show that $X_C^P(\mu_P) = S^3$, for $\mu_P \subset \partial X_P = \partial X_C^P$ the meridian of $K_P \subset S^3$ as an oriented knot.

(b) Let $X_{U,P}$ be the exterior $X_{U,P} := S^3 \setminus (\nu(U) \amalg \nu(K_P))$, and let $\iota_*^{X_{U,P}}$, $\iota_*^{\partial X_U}$, and $\iota_*^{\partial X_P}$ be the homomorphisms induced on first homology by the respective inclusions

$$\iota^{X_{U,P}} : X_{U,P} \to X_P, \quad \iota^{\partial X_U} : \partial X_U \to X_P, \quad \iota^{\partial X_P} : \partial X_P \to X_P,$$

where $\iota^{\partial X_U}$ is the composition of the inclusion $\partial X_U = \partial_U X_{U,P} \hookrightarrow X_{U,P}$ with $\iota^{X_{U,P}}$. Show that

$$(*) \qquad \iota_*^{X_{U,P}} \tau(X_{U,P}) = \frac{\Delta(X_P)(1 - \iota_*^{\partial X_U}[\lambda_U])}{1 - \iota_*^{\partial X_P}[\mu_P]}$$

holds, where τ and Δ denote Turaev torsion and Alexander polynomial, respectively.

(c) Let $K_C^{K_P} := \operatorname{core}(X_C^P(\mu_P) \setminus X_C^P) \subset S^3$ be the core knot of the Dehn filling $X_C^P(\mu_P) \cong S^3$ of X_C^P; in other words, $K_C^{K_P}$ is the satellite of K_C by K_P. Let $\iota_*^C : H_1(X_C; \mathbb{Z}) \to H_1(X_C^P; \mathbb{Z})$ be the homomorphism induced by the inclusion $\iota_C : X_C \to X_C^P$. Show that there is a canonical isomorphism $\phi : H_1(X_P) \to H_1(X_C^P)$ such that

$$\Delta_{K_C^{K_P}} = \iota_*^C(\Delta_{K_C})\phi(\Delta_{K_P}),$$

where $\Delta_K \in \mathbb{Z}[H_1(X_K; \mathbb{Z})]$ denotes the Alexander polynomial of a knot $K \subset S^3$ with exterior $X_K := S^3 \setminus \nu(K)$.

(d) Suppose, for relatively prime integers $p, q > 1$, that $K_P \subset S^3$ is the (p, q) torus knot $K_P = T_{p,q} \subset S^3$, such that $T_{p,q}$ is isotopic in X_U to a curve of slope p/q in ∂X_U. The satellite $K_C^{(p,q)} := K_C^{T_{p,q}}$ is called the (p, q)-*cable* of K_C. Compute the Alexander polynomial $\Delta_{T_{3,2}^{(4,3)}}$ of the $(4, 3)$-cable of $T_{3,2}$. You do not need to simplify your answer. You may use any results from lectures or example sheets, including the fact that up to multiplication by units, the (p, q) torus knot $T_{p,q} \subset S^3$ has Alexander polynomial

$$\frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}.$$

END OF PAPER