UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2019 9:00 am to 12:00 pm

PAPER 138

MODULAR REPRESENTATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let A be a finite-dimensional algebra over a field k. Let M be a finite-dimensional A-module. What does it mean to say that M is *semisimple*? What does it mean to say that A is *semisimple*?

In the following, $J(M)$ denotes the radical of M (the intersection of all maximal submodules of M), and $\mathrm{Soc}(M)$ is the socle of M (the largest semisimple submodule of M). You may use any appropriate equivalent descriptions of $J(M)$ and $\mathrm{Soc}(M)$, without proof, provided they are clearly stated.

Write down the classification of the indecomposable modules for a cyclic group of order p^n over a field of characteristic p. Identify the radical and socle of each indecomposable module. (Only brief justifications are required.)

(b) Let A be a finite-dimensional algebra over a field k. For each A-module M, define inductively $J^n(M) := J(J^{n-1}(M))$ and $\mathrm{Soc}^n(M)/\mathrm{Soc}^{n-1}(M) := \mathrm{Soc}(M/\mathrm{Soc}^{n-1}M)$. These submodules form chains, $\cdots \subseteq J^2(M) \subseteq J(M) \subseteq M$ and $0 \subseteq \mathrm{Soc}(M) \subseteq \mathrm{Soc}^2(M) \subseteq \cdots$, called the *radical series* and *socle series*, respectively.

Let U and V be finite-dimensional A-modules. Show that for each n, $J^n(U \oplus V) = J^n(U) \oplus J^n(V)$ and $\mathrm{Soc}^n(U \oplus V) = \mathrm{Soc}^n(U) \oplus \mathrm{Soc}^n(V)$.

Deduce that if k is a field of characteristic p and G is a p-group then the regular representation, kG, is indecomposable.

(c) Let A be a finite-dimensional algebra over a field k, and let M be an A-module. Show that the radical series of M is the fastest descending series of submodules of M with semisimple quotients, and the socle series of M is the fastest ascending series of M with semisimple quotients. Show that the two series terminate, and if m and n are the least integers for which $J^m(M) = 0$ and $\mathrm{Soc}^n(M) = M$, then $m = n$.

(d) Let $A = T_n(k)$ be the algebra of $n \times n$ lower triangular matrices.

(i) Construct a set of n non-isomorphic simple A-modules, $S_1, \dots, S_n$. Also find $J(A)$.

(ii) Determine the radical series for the regular A-module ${}_AA$.

(iii) Using an appropriate ordering of the simple modules appearing in (i), show that $J^i(A)/J^{i+1}(A) \cong S_{i+1} \oplus \cdots \oplus S_n$.

2

(a) Let G be a finite group and let k be a field of characteristic p. Assume k has all $|G|_{p'}$-th roots of unity. If g is a p'-element of G and M is a finite-dimensional kG-module, define the *Brauer character* $\chi_M(g)$ of M. Show that the irreducible Brauer characters χ_{S_i} are linearly independent over $\mathbb{C}$.

(b) Let $(K, \mathfrak{O}, k)$ be a p-modular system where $\mathfrak{O}$ is complete, and let G be a finite group. Suppose that P and U are finite-dimensional kG-modules and that P is projective. Show that

$$\dim \operatorname{Hom}_{kG}(P,U) = \frac{1}{|G|} \sum \chi_P(g^{-1})\chi_U(g)$$

where the sum is over all p'-elements $g \in G$. [You may wish to begin by considering the isomorphism $\operatorname{Hom}_{kG}(M,N) \cong \operatorname{Hom}_{kG}(M \otimes_k N^*, k)$ whenever M, N are finite-dimensional kG-modules.]

(c) Let G be a finite group and suppose that k is a splitting field for G of characteristic p. Let $\mathfrak{C}_{p'}$ be the vector space of class functions from the p'-classes to $\mathbb{C}$, endowed with the Hermitian inner product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum \overline{\phi(g)}\psi(g)$$

summed over all p'-elements $g \in G$. Let $S_1, \ldots, S_n$ be a complete list of non-isomorphic simple kG-modules, with projective covers $P_1, \ldots, P_n$.

The Brauer characters $\chi_{S_1}, \ldots, \chi_{S_n}$ of the simple modules form a basis for $\mathfrak{C}_{p'}$, as do also the Brauer characters $\chi_{P_1}, \ldots, \chi_{P_n}$ of the indecomposable projective modules (you are not asked to prove this). Show that these two bases are dual to each other with respect to the bilinear form, in the sense that

$$\langle \chi_{P_i}, \chi_{S_j} \rangle = \delta_{i,j}.$$

Why is the bilinear form on $\mathfrak{C}_{p'}$ non-degenerate?

(d) With the preceding notation, let X denote the table of Brauer character values of simple kG-modules, let Π be the table of Brauer character values of indecomposable projective modules, and let D be the diagonal matrix whose entries are $1/|C_G(x_i)|$, as x_i ranges through representatives of the p'-classes.

Show that $\bar{\Pi} D X^T = I$ and deduce that $\bar{X}^T \Pi$ is the diagonal matrix with entries $|C_G(x_i)|$, where the x_i are representatives of the p'-classes of elements of G. Hence show that

$$\sum_{\text{simple } S} \chi_S(g^{-1})\chi_{P_S}(h) = |C_G(g)|$$

if g and h are conjugate, otherwise 0.

3

(a) Let k be a field and G a finite group. Show that $kG^* \cong kG$ as kG-modules. Deduce that a finitely-generated kG-module P is projective if and only if P^* is projective as a kG-module.

Show that finitely-generated projective kG-modules are the same as finitely-generated injective kG-modules. Show also that each indecomposable projective kG-module has a simple socle.

(b) Let P be an indecomposable projective module for a group algebra kG. Show that $P/J(P) \cong \mathrm{Soc}(P)$.

(c) For any kG-module M, define the *fixed points* of G on M to be $M^G := \{m \in M : gm = m \text{ for all } g \in G\}$. Define the *fixed quotient* of G on M to be $M_G := M/\langle (g-1)m : m \in M,\ 1 \neq g \in G\rangle$. Evidently M^G is the largest submodule of M on which G acts trivially and M_G is the largest quotient of M on which G acts trivially.

If P is any projective kG-module and S is a simple kG-module, show that the multiplicity of S in $P/J(P)$ equals the multiplicity of S in $\mathrm{Soc}(P)$. Deduce that

$$\dim P^G = \dim P_G = \dim (P^*)^G = \dim (P^*)_G.$$

In the usual notation for projective covers, show also that for every simple kG-module S, $(P_S)^* \cong P_{S^*}$.

(d) All modules in this part are to be finite-dimensional. Let M be an indecomposable kG-module, where k is a field, and let P_k be the projective cover of the trivial module. Prove that

$$\dim((\sum_{g \in G} g) \cdot M)$$

is 1, if $M \cong P_k$, otherwise it is 0. [Hint: first observe that $(kG)^G = P_k^G = k \cdot \sum_{g \in G} g$. You may wish to recall that P_k is injective and has socle isomorphic to k.]

4

(a) Let R be a commutative ring with 1. Let G be a finite group and let H be subgroup of G. In terms of commutative diagrams, what does it mean for an RG-module to be *(relatively) H-projective*? State D. Higman's criterion for relative projectivity of a module. In what follows you may assume any equivalent definitions of relative projectivity without proof.

(b) (i) If H is a subgroup of G and the index $|G:H|$ is invertible in the ring R, show that every RG-module is H-projective.

(ii) Suppose that H is a subgroup for which $|G:H|$ is invertible in the ring R, and let M be an RG-module. Show that M is projective as an RG-module if and only if $M\downarrow_H^G$ is projective as an RH-module.

(c) Let $G = \mathrm{SL}_2(p)$. You may assume that the symmetric powers $S^r(V_2)$ are all the simple $\mathbb{F}_pG$-modules when $0 \leqslant r \leqslant p-1$, where V_2 is the 2-dimensional space on which G acts as invertible transformations of determinant 1. Show that on restriction to the Sylow p-subgroup of upper unitriangular matrices, $S^r(V_2)$ is indecomposable of dimension $r+1$ when $0 \leqslant r \leqslant p-1$. Deduce the existence of an ordinary irreducible character of G of degree p. [The classification of indecomposable modules for cyclic groups of prime order may be assumed.]

(d) Recall that a ring A has *finite representation type* if and only if there are only finitely many isomorphism classes of indecomposable A-modules. Let k be a field of characteristic p and let P be a Sylow p-subgroup of a finite group G. Show that kG has finite representation type if and only if kP has finite representation type [the Krull–Schmidt theorem may be assumed]. Show further that kG has finite representation type if and only if Sylow p-subgroups of G are cyclic. [If you wish, you may assume that if $G = C_p \times C_p$ then there are infinitely many non-isomorphic indecomposable kG-modules.]

(e) Let M be an indecomposable kG-module (k a field). Show that there is a unique conjugacy class of subgroups Q of G that are minimal with respect to the property that M is Q-projective (Mackey's restriction formula may be used, if required). Any such Q is a called a *vertex*. Explain why the vertex of any indecomposable module is a p-group. Find the vertex of the trivial kG-module, k_G.

5

(a) Let R be a finite-dimensional algebra. What does it mean to say that the R-module M *lies in the block (idempotent)* e? If e is a block of R and if $0 \to U \to V \to W \to 0$ is a short exact sequence of R-modules, show that V belongs to e if and only if U and W belong to e.

(b)Let R be either $\mathfrak{O}$ a discrete valuation ring with residue field of characteristic p or k, a field. Define the *defect group* of a block of RG and show the defect groups of a block are all conjugate.

(c) Consider the simple group G=GL$_3(2)$ of 3×3 nonsingular matrices over $\mathbb{F}_2$. You may assume from group theory that G has order $168 = 8.3.7$ and that G has six conjugacy classes. Its ordinary character table is given below. The numbers that label the conjugacy classes of elements in the top row indicate the order of the elements:

g	1	2	4	3	$7A$	$7B$
$\lvert C_G(g)\rvert$	168	8	4	3	7	7
	1	1	1	1	1	1
	3	-1	1	0	α	$\bar{\alpha}$
	3	-1	1	0	$\bar{\alpha}$	α
	6	2	0	0	-1	-1
	7	-1	-1	1	0	0
	8	0	0	-1	1	1

where $\alpha = \eta + \eta^2 + \eta^4$ and $\eta = e^{2\pi i/7}$.

(i) Let k be a splitting field for G of characteristic 2. Compute the table of Brauer characters of simple kG-modules. Identify any blocks of defect 0. [Hint: show that the natural 3-dimensional space of column vectors on which G acts is a 2-modular simple representation.]

(ii) Find the decomposition matrix and Cartan matrix of G for the prime 2.

(iii) Write down the table of Brauer characters of projective indecomposable kG-modules.

(iv) Express $8 \otimes 3$ as a direct sum of indecomposable modules (where 8 and 3 denote the simple kG-modules of those dimensions). [Hint: show that $8 \otimes 3$ is projective, hence is a direct sum of projective indecomposable modules; then consider inner products of characters.]

6

(a) Let $(K, \mathfrak{O}, k)$ be a splitting p-modular system such that $\mathfrak{O}$ is complete, and let G be a group of order $p^d q$ where q is prime to p. Let M be a KG-module of dimension n, containing an $\mathfrak{O}$-form W. Write $\bar{W} = k \otimes_{\mathfrak{O}} W = W/\mathfrak{p}W$ for the p-modular reduction. Consider the five statements below, and prove the four implications (ii) $\Longrightarrow$ (iii), (ii) $\Longrightarrow$ (iv), (iii) $\Longrightarrow$ (v) and (v) $\Longrightarrow$ (i).

(i) p^d divides n and M is a simple KG-module.

(ii) The homomorphism $\mathfrak{O}G \to \mathrm{End}_{\mathfrak{O}}(W)$ that gives the action of $\mathfrak{O}G$ on W identifies $\mathrm{End}_{\mathfrak{O}}(W) \cong M_n(\mathfrak{O})$ with a ring direct factor of $\mathfrak{O}G$.

(iii) M is a simple KG-module and W is a projective $\mathfrak{O}G$-module.

(iv) The homomorphism $kG \to \mathrm{End}_k(\bar{W})$ identifies $\mathrm{End}_k(\bar{W}) \cong M_n(k)$ with a ring direct factor of kG.

(v) As a kG-module $\bar{W}$ is simple and projective.

(b) Let k be a field of characteristic p that is a splitting field for G and all of its subgroups and let D be a p-subgroup of G.

(i) Define the *Brauer morphism*, explaining all terms appearing in your definition.

(ii) Show that the following diagram commutes:

$$\begin{array}{ccc} (kG)^D & \stackrel{\beta}{\to} & kC_G(D) \\ \downarrow\tau & & \downarrow\tau' \\ (kG)^G_D & \stackrel{\beta'}{\to} & (kC_G(D))^{N_G(D)}_D \end{array}$$

where β, β' are appropriate Brauer morphisms and τ, τ' are appropriate transfer maps.

(c) Consider the group $G = A_5$ and let k be a splitting field of characteristic 5. Then X, the 5-modular simple Brauer character table and Y, the decomposition matrix of G at the prime 5 are given below with obvious notation (you are not asked to verify these).

$X =$

	1	(12)(34)	(123)
ϕ_1	1	1	1
ϕ_2	3	-1	0
ϕ_3	5	1	-1

$Y =$

	ϕ_1	ϕ_2	ϕ_3
χ_1	1	0	0
χ_{3A}	0	1	0
χ_{3B}	0	1	0
χ_4	1	1	0
χ_5	0	0	1

For each defect group D, find the normaliser N, and describe the Brauer correspondence between certain blocks of G and certain blocks of N.

END OF PAPER