UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 9:00 am to 12:00 pm

PAPER 123

ALGEBRAIC NUMBER THEORY

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let R be a Dedekind domain with fraction field K, $L \supseteq K$ a finite separable extension, $B \supseteq R$ the integral closure of R in L.

(i) Fix a non-zero prime ideal q of B, and let $p = q \cap R$. Define the ramification index $e_{q/p}$, the residue degree $f_{q/p}$, and show that for fixed p,

$$[L : K] = \sum_{q|p} e_{q/p} f_{q/p}.$$

(ii) Let $\widehat{R_p}$ be the completion of R at p, K_p its fraction field, $\widehat{B_q}$ be the completion of B at q, and L_q its fraction field.

Describe the relationship between the ramification indices and residue extensions of $\widehat{B_q}$ and of B. [You need not prove your answer.]

(iii) Now suppose the extension L/K is Galois. Define the inertia group $I_{q/p}$, the decomposition group $D_{q/p}$ and describe the relation between these groups, $Gal(L/K)$ and $Gal(L_q/K_p)$.

(iv) Let $R = \mathbb{Z}$, and let L be the field generated by a root of the cubic polynomial $x^3 + 25x^2 - 50x + 40$.

Describe L_q, a uniformiser for it, and its residue field, for all primes q lying over the primes $p = 2$ and $p = 5$.

Let E be the Galois closure of $L/\mathbb{Q}$.

Determine the decomposition group and inertia groups for $p = 2, 5$ in $Gal(E/\mathbb{Q})$.

Determine $Gal(E/\mathbb{Q})$.

[The prime factorisation of the discriminant of this polynomial is $-4 \times 25 \times 13807$.]

2

(i) State Krasner's lemma, and using it show that the completion $\mathbb{C}_p$ of the algebraic closure of $\mathbb{Q}_p$ is algebraically closed.

(ii) Determine the radius of convergence of the power series $exp(x) = \sum_{n \geqslant 0} \frac{x^n}{n!}$ as a function on $\mathbb{C}_p$.

3

(i) Let $L \supseteq K$ be an extension of local fields. Define what it means for the extension to be unramified.

Now let $L \supseteq K$ be an extension of number fields. Define what it means for the extension to be unramified *everywhere*.

(ii) Let K be a number field. Define the Hilbert class field $\tilde{K}$.

(iii) Compute the Hilbert class field when $K = \mathbb{Q}(\sqrt{-6})$.

4

Let K be a complete local field, $\mathcal{O}$ the ring of integers in K, $\mathfrak{m}$ the maximal ideal, π a uniformiser, $k = F_q$ the residue field, and suppose *char* $K = 0$.

(i) Prove $K^* \cong \mathbb{Z} \times \mu_{q-1} \times (1 + \mathfrak{m})$, stating clearly any results you use. Describe the structure of $(1 + \mathfrak{m})$ as an abelian group [you need not prove your answer].

(ii) State the main theorem of local class field theory, describing the finite abelian extensions of K.

(iii) Compute, for all $n > 0$, the subgroup $N_{\mathbb{Q}_2(\xi_{2^n})/\mathbb{Q}_2}(\mathbb{Q}_2(\xi_{2^n})^*)$, where ξ_{2^n} is a primitive 2^n'th root of unity.

END OF PAPER