UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 1:30 pm to 4:30 pm

PAPER 121

TOPICS IN SET THEORY

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

Questions 1 and 2 each carry 30 marks. Question 3 carries 40 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Determine for each of the following concepts whether they are "upwards absolute but not downwards absolute" (**U**), "downwards absolute but not upwards absolute" (**D**), "both upwards and downwards absolute" (**U+D**), or "neither upwards nor downwards absolute" (**N**) between transitive models of ZFC. [*No proofs are needed.*]

(a) "x is infinite";

(b) "x is a cardinal";

(c) "x is a partial order";

(d) "x is a subset of $\mathbb{N}$";

(e) "x is countable"; and

(f) "x is a cardinal with $\mathrm{cf}(x) < x$".

(ii) Prove that if α is a countable ordinal, then $\mathbf{V}_\alpha \not\models \mathsf{ZFC}$.

(iii) Let M be a uncountable transitive set model of ZFC. Show that M must contain uncountably many ordinals.

(iv) Let $T \supseteq \mathsf{ZFC}$ be any complete extension of the axioms of ZFC in the language of set theory, i.e., for all sentences ψ, either $\psi \in T$ or $\neg\psi \in T$. We say that a set model $M \models T$ is a *Paris model* if every ordinal in M is definable, i.e., for every $\alpha \in M$ such that $M \models$"α is an ordinal", there is a formula φ such that $M \models \forall z(z = \alpha \leftrightarrow \varphi(z))$. A model of set theory is called *rigid* if every automorphism is the identity. Prove the following statements:

(a) If there is a wellfounded model of T, then all Paris models of T are wellfounded.

(b) If there is a wellfounded model of T, then all Paris models of T are rigid.

2

In the following, work under the assumption that there is a transitive set model M of ZFC.

(i) Let $A \in M$. Give the definition of $\mathcal{D}(A)$, the *definable power set of* A.

[*You may assume that* $\mathrm{Def}(A, n) \subseteq A^n$ *is already defined.*]

(ii) Let $\mathbf{L}$ be the constructible universe inside M. Show that $\mathbf{L}$ satisfies the power set axiom.

(iii) Explain why in general the set $\mathcal{D}(A)$ is not the constructible power set of A.

[*You may use basic properties of the constructible hierarchy in your argument if you state them explicitly.*]

(iv) For any set $A \in M$, let $\mathbf{L}(A)$ be the constructible universe built from $\mathrm{tcl}(A)$ inside M. Show that the following statement does *not* in general hold: if $A \subseteq \omega_2$, then $\mathbf{L}(A) \models \mathsf{CH}$.

[*In this argument, you may use the consistency results from forcing proved in class, e.g., that there is a transitive model of* $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_2$.]

3 In this question, work under the assumption that M is a countable transitive model of ZFC.

(i) Let $p \in \mathbb{P} \in M$. Define the notions

(a) *D is dense below p,*

(b) *G is a $\mathbb{P}$-generic filter over M*, and

(c) *$\mathbb{P}$ has the κ-chain condition.*

(ii) Prove that if $M \models$"$\mathbb{P}$ has the κ-chain condition" and $M \models$"κ is regular", then $\mathbb{P}$ preserves cardinals $\geqslant \kappa$.

(iii) Let $\Vdash$ be the semantic forcing relation and $\Vdash^*$ be the syntactic forcing relation. Let φ be a sentence in the forcing language. Show that $p \Vdash \varphi$ if and only if $p \Vdash^* \varphi$.

[*You may use the Forcing Theorem without proof and you may also assume the following equivalence: $p \Vdash^* \varphi$ if and only if $\{r\,;\, r \Vdash^* \varphi\}$ is dense below p if and only if for every $r \leqslant p$, $r \Vdash^* \varphi$.*]

(iv) Let $\mathbb{P} \in M$ and G be $\mathbb{P}$-generic over M. Show that $M[G] \models \mathsf{Separation}$.

(v) Let $\kappa \in M$ be a cardinal in M. Define $\mathbb{P} := \{p\,;\, p$ is a partial function with $\mathrm{dom}(p) \subseteq \kappa \times \mathbb{N}$ finite, $\mathrm{ran}(p) \subseteq \kappa$, and if $(\alpha, n) \in \mathrm{dom}(p)$, then $p(\alpha, n) < \alpha\}$. Let G be $\mathbb{P}$-generic over M.

(a) Show that if $\lambda < \kappa$ is a cardinal in M, then λ is countable in $M[G]$.

(b) Show that if κ is regular in M, then $\mathbb{P}$ has the κ-chain condition.

(c) Show that κ is regular in M if and only if $M[G] \models \kappa = \aleph_1$.

END OF PAPER