UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2019 9:00 am to 12:00 pm

PAPER 119

CATEGORY THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the notions of *regular epimorphism* and *strong epimorphism*, and show that every regular epimorphism is strong.

Let $\mathcal{C}$ be a category with pullbacks and coequalizers, in which any pullback of a regular epimorphism is epic. Show that every morphism of $\mathcal{C}$ factors as a regular epimorphism followed by a monomorphism, and deduce that every strong epimorphism is regular.

Let **Cat** denote the category of small categories and functors between them. Give, with brief justification, an example of a strong epimorphism in **Cat** which is not regular. Does every morphism in **Cat** factor (*a*) as a regular epi followed by a mono, or (*b*) as a strong epi followed by a mono? Justify your answers.

[You may assume without proof the result that monomorphisms are always stable under pullback, and the fact that monomorphisms in **Cat** are injective functors.]

2

Explain what is meant by a *congruence* on a category $\mathcal{C}$, and by the *quotient category* $\mathcal{C}/\sim$ where $\sim$ is a congruence on $\mathcal{C}$.

Let $\mathcal{C}$ be a category with finite products, and let Φ be a filter of subobjects of the terminal object 1 of $\mathcal{C}$ (i.e. a family such that $1 \in \Phi$, $(U, V \in \Phi \Rightarrow U \times V \in \Phi)$ and $(U \in \Phi, U \leqslant V \Rightarrow V \in \Phi)$). For objects A and B of $\mathcal{C}$, we define a Φ-*map* $A \to B$ to be an equivalence class of morphisms $A \times U \to B$ with $U \in \Phi$, where two such morphisms $A \times U \to B$ and $A \times V \to B$ are equivalent iff there exists $W \in \Phi$ with $W \leqslant U \times V$ such that

$$\begin{array}{ccc} A \times W & \longrightarrow & A \times V \\ \downarrow & & \downarrow \\ A \times U & \longrightarrow & B \end{array}$$

commutes. Verify that this is indeed an equivalence relation, and that the objects and Φ-maps of $\mathcal{C}$ form a category $\mathcal{C}_\Phi$.

We define a functor $P_\Phi \colon \mathcal{C} \to \mathcal{C}_\Phi$ by setting $P_\Phi(A) = A$, and taking $P_\Phi(f \colon A \to B)$ to be the equivalence class of $A \times 1 \cong A \xrightarrow{f} B$. Show that $\mathcal{C}_\Phi$ has and P_Φ preserves finite products. If $\mathcal{C}$ is cartesian closed, show also that $\mathcal{C}_\Phi$ is cartesian closed and P_Φ preserves exponentials.

Is $\mathcal{C}_\Phi$ (isomorphic to) a quotient of $\mathcal{C}$ by a congruence? Justify your answer.

3

Explain briefly how an adjunction between functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ may be specified by a suitable pair of natural transformations $\eta: 1_{\mathcal{C}} \to GF$ and $\epsilon: FG \to 1_{\mathcal{D}}$.

Let $F_1, F_2 : \mathcal{C} \rightrightarrows \mathcal{D}$ be functors having right adjoints G_1, G_2 respectively. Show that there is a bijection between natural transformations $\alpha : F_1 \to F_2$ and natural transformations $\overline{\alpha}: G_2 \to G_1$.

Now suppose $F : \mathcal{C} \to \mathcal{C}$ carries a monad structure $\mathbb{F}$ and also has a right adjoint G. Show that G carries a comonad structure $\mathbb{G}$ for which the category of $\mathbb{G}$-coalgebras is isomorphic to the category of $\mathbb{F}$-algebras. Deduce that if M is a monoid then the forgetful functor $[M, \mathbf{Set}] \to \mathbf{Set}$ is comonadic.

[You may assume that the bijection constructed in the second part is contravariantly functorial, in the sense that if (α, β) is a composable pair of natural transformations between left adjoints then $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$, and also that $\overline{L\alpha} = \overline{\alpha}_R$ if $(L \dashv R)$.]

4

Explain briefly what is meant by the *monad* induced by an adjunction, and by an *algebra* for a monad.

Given an adjunction $(F: \mathcal{C} \to \mathcal{D} \dashv G: \mathcal{D} \to \mathcal{C})$ inducing a monad $\mathbb{T}$ on $\mathcal{C}$, define the Eilenberg–Moore comparison functor $K: \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$, and show that if $\mathcal{D}$ has coequalizers of reflexive pairs then K has a left adjoint L. Explain what is meant by the *monadic length* of such an adjunction $(F \dashv G)$.

Let $\mathcal{C}_0 = \mathbf{Set}$, and for $n > 0$ let $\mathcal{C}_n$ be the category of sets A equipped with n partial unary operations $\alpha_1, \alpha_2, \ldots, \alpha_n : A \rightharpoonup A$ such that $\alpha_1(a)$ is defined for all $a \in A$, and for $i > 1$ $\alpha_i(a)$ is defined if and only if ($\alpha_{i-1}(a)$ is defined and) $\alpha_{i-1}(a) = a$. Show that the forgetful functor $\mathcal{C}_{n+1} \to \mathcal{C}_n$ has a left adjoint for any n [*hint: the free $\mathcal{C}_{n+1}$-object on a $\mathcal{C}_n$-object A may be taken to have underlying set $A \times \{0\} \cup A_n \times \mathbb{N}$, where $A_n = \{a \in A \mid \alpha_n(a) = a\}$*.] Show also that, for any $m < n$, the monad on $\mathcal{C}_m$ induced by the composite adjunction $\mathcal{C}_m \rightleftarrows \mathcal{C}_n$ coincides with that induced by $\mathcal{C}_m \rightleftarrows \mathcal{C}_{m+1}$. Assuming the result that the latter adjunction is monadic, deduce that $\mathcal{C}_0 \rightleftarrows \mathcal{C}_n$ has monadic length n.

5

Let $\mathcal{C}$ be a category with finite products. What does it mean to say that an object of $\mathcal{C}$ is *exponentiable*? Show that the class of exponentiable objects of $\mathcal{C}$ is closed under finite products. Show also that if $\mathcal{C}$ has an object which is both terminal and initial, then it is (up to isomorphism) the only exponentiable object.

Recall that a topological space X is said to satisfy the T_0 *axiom* if, for any two distinct points x, y of X, there is an open set containing just one of x and y. Show that any T_0-space X is homeomorphic to a subspace of a power $\prod_{g \in G} S$ of the *Sierpiński space* S, i.e. the set $\{0,1\}$ with $\{1\}$ open but $\{0\}$ not open. [*Hint: take G to be the set of continuous maps $X \to S$.*] Deduce that there is an equalizer diagram

$$X \rightarrowtail \prod_{g \in G} S \rightrightarrows \prod_{h \in H} S$$

for some sets G and H.

Hence show that a space E is exponentiable in the full subcategory $\mathbf{Top}_0 \subseteq \mathbf{Top}$ of T_0-spaces iff the functor $\mathbf{Top}_0\,(- \times E, S)$ is representable.

[You may assume that $\mathbf{Top}_0$ is closed under products in $\mathbf{Top}$, and that its regular monomorphisms are subspace inclusions.]

6

State the Yoneda Lemma (including the 'naturality' assertion). Deduce that the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is full and faithful, for any locally small category $\mathcal{C}$.

Sketch the proof that $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos when $\mathcal{C}$ is small. Hence show that

(a) every small category with finite limits is equivalent to a full subcategory of a topos closed under finite limits;

(b) every small cartesian closed category is equivalent to a full subcategory of a topos closed under finite products and exponentials;

(c) every small category with finite products and splittings of idempotents is equivalent to the full subcategory of tiny objects of a topos, i.e. objects A such that $(-)^A$ has a right adjoint.

[For (c), you may assume the result that if $\mathcal{C}$ has splittings of idempotents then the representable functors are exactly the objects F of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ for which $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(F, -)$ preserves all small colimits.]

END OF PAPER